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A Family of High Continuity Subdivision Schemes Based on Probability Distribution

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ABSTRACT

Subdivision schemes are famous for the generation of smooth curves and surfaces in CAGD (Computer Aided Geometric Design). The continuity is an important property of subdivision schemes. Subdivision schemes having high continuity are always required for geometric modeling. Probability distribution is the branch of statistics which is used to find the probability of an event. We use probability distribution in the field of subdivision schemes. In this paper, a simplest way is introduced to increase the continuity of subdivision schemes. A family of binary approximating subdivision schemes with probability parameter p is constructed by using binomial probability generating function. We have derived some family members and analyzed the important properties such as continuity, Holder regularity, degree of generation, degree of reproduction and limit stencils. It is observed that, when the probability parameter $p = 1/2$, the family of subdivision schemes have maximum continuity, generation degree and Holder regularity. Comparison shows that our proposed family has high continuity as compare to the existing subdivision schemes. The proposed family also preserves the shape preserving property such as convexity preservation. Subdivision schemes give negatively skewed, normal and positively skewed behavior on convex data due to the probability parameter. Visual performances of the family are also presented.

Key Words: Binary, Approximating Subdivision Schemes, Binomial Distribution, Continuity Analysis, Convexity Preservation.

1. INTRODUCTION

CAGD concerns with the mathematical description of shape for use in computer graphics. Subdivision schemes are iterative formulas for generation of smooth curves and surfaces. They give effective approaches to generate smooth curves and surfaces from discrete arrangement of control points.

Initially, De Rham [1] gave the idea of CAGD with geometry. After that Chaikin [2] introduced a procedure to generate curves from limited number of points. This algorithm was one of the first refinement algorithms to generate curves. Dyn et. al. [3] presented a subdivision scheme with tension parameter which is C^1 continuous for a certain range of parameter. Mustafa and Liu [4] presented a new solid parametric subdivision scheme.

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Mustafa et. al. [5] proposed and analyzed the *m*-point binary approximating parametric subdivision scheme and compared the proposed scheme with existing schemes. Ghaffar and Mustafa [6] introduced new families of subdivision schemes for curve design. Mustafa et. al. [7] presented a family of (2*n*-1)-point binary approximating subdivision schemes with free parameters for describing curves. Ashraf et. al. [8] presented a new family of subdivision schemes by using a variant of Lane-Riesenfeld algorithm.

Cao and Tan [9] presented and analyzed the important properties of new 5-point relaxed subdivision scheme. Their scheme produce C^2 limit curve when $w \in \left(\frac{-3}{64}, \frac{1}{162}\right)$.

Mustafa et. al. [10] presented a proof of 6-point scheme. They said that the scheme produce C^2 and C^3 limit curve when $w \in [0,0425]$, and $w \in [0.0139, 0.0143]$, respectively.

Bari et. al. [11] presented shape preserving subdivision schemes. Siddiqi and Noreen [12] presented the convexity preserving property of 6-point ternary interpolating subdivision scheme with the tension parameter. Tan et. al. [13] also discussed the convexity preserving property of 5-point binary scheme.

Zheng et. al. [14] presented a technique to increase the continuity of any subdivision scheme. They multiply the symbol of the scheme with $(1+z/2)^k$ factor to get C^{n+k} continuous subdivision schemes. But the technique used by them increase the complexity of the scheme. In this paper, we have presented a way to introduce a new family of subdivision schemes. Our technique is based on probability generating function of Binomial probability distribution. By using this technique one can able to increase the continuity, Holder regularity, degree of generation, degree of reproduction of any subdivision scheme without increasing the complexity of the scheme.

Probability is the chance of occurrence. A variable that shows the probabilities as the outcome of an experiment is called random variable. A list of probabilities associated to each value of random variable is called probability distribution and a function that is used for this purpose is called probability generating function. Binomial probability distribution was introduced by Bernoulli [15].

The paper is organized as follows. In Section 2, we construct a general symbol of family of binary approximating subdivision scheme. Complete analysis of some family members of proposed family is presented in Section 3. Convexity preservation is discussed in Section 4. Applications and comparison are presented in Section 5. Conclusions are drawn in Sections 6.

2. CONSTRUCTION OF A FAMILY OF SCHEMES

In this section, we will present a generalized symbol for the construction of a family of binary approximating subdivision schemes. For this, we consider a well-known 4-point scheme [16] given by:

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{384}\left(27f_{i-1}^{k} + 235f_{i}^{k} + 121f_{i+1}^{k} + f_{i+2}^{k}\right) \\ f_{2i+1}^{k+1} &= \frac{1}{384}\left(f_{i-1}^{k} + 121f_{i}^{k} + 235f_{i+1}^{k} + 27f_{i+2}^{k}\right) \end{aligned} \tag{1}$$

The Laurent polynomial of the scheme Equation (1) is:

$$B(z) = \frac{1}{384}\left(1 + 27z + 121z^2 + 235z^3 + 235z^4 + 121z^5 + 27z^6 + z^7\right) \tag{2}$$

Binomial probability generating function [17] is given as:

$$H_n(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i \tag{3}$$

where $i = 0,1,2,\ldots,n$.

Here n is the number of trails, p is the probability of success, $q = 1 - p$ is the probability of failure and also $p + q = 1$. For the construction of a family of binary approximating subdivision schemes, we multiply $B(z)$ and $H_n(z)$.

$$A_n(z) = \frac{1}{384}(1+z)^5\left(1+22z+z^2\right)\sum_{i=0}^{n}\binom{n}{i} p^i (1-p)^{n-i} z^i \tag{4}$$

We can easily derive symbol of the schemes by substituting = 1,2,…, in Equation (4) with $p \in (0,1)$. Before going onward to elaborate behavior and considerable properties of the proposed family of schemes $A_n = \{A_n : n = 1,2,\ldots\}$, let us have a smart look at some members of this family.

2.1 DERIVATION OF FAMILY MEMBERS

Relaxed 4-Point Scheme (i.e. A_1): After substituting $n = 1$ in Equation (4), we get the following symbol of relaxed 4-point binary approximating subdivision scheme

$$\begin{aligned} A_1(z) = \frac{1}{384}\big[&(1+p)+(27-26p)z+(121-94p)z^2+(235-114p)z^3+235z^4 \\ &+(121+114p)z^5+(27+94p)z^6+(1+26p)z^7+pz^8\big] \end{aligned} \tag{5}$$

The scheme corresponding to Equation (5) is:

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{384}\left[(27-26p)f_{i-2}^k+(235-114p)f_{i-1}^k+(121+114p)f_i^k+(1-26p)f_{i+1}^k\right] \\ f_{2i+1}^{k+1} &= \frac{1}{384}\left[(1-p)f_{i-2}^k+(121-94p)f_{i-1}^k+235f_i^k+(27+94p)f_{i+1}^k+pf_{i+1}^k\right] \end{aligned} \tag{6}$$

5-Point Scheme (i.e. A_2): By putting $n = 2$ in Equation (4), we obtain the symbol of 5-point binary approximating subdivision scheme

$$A_2(z) = \frac{1}{384}(1+z)^5\left(1+22z+z^2\right)\left(p^2+2p\right)(1-p)z+(1-p)^2z^2 \tag{7}$$

The scheme corresponding to Equation (7) is:

$$\begin{aligned} f_{2i}^{k+1} = \frac{1}{384}\big[&\left(25p^2+2p\right)f_{i-2}^k+\left(20p^2+188p+27\right)f_{i-1}^k+\left(-114p^2+235\right) \\ &f_i^k+\left(68p^2-188p+121\right)f_{i+1}^k+\left(p^2-2p+1\right)f_{i+2}^k\big] \end{aligned}$$

$$\begin{aligned} f_{2i+1}^{k+1} = \frac{1}{384}\big[&p^2f_{i-2}^k+\left(68p^2+52p+1\right)f_{i-1}^k+\left(-114p^2+228p+121\right) \\ &f_i^k+\left(20p^2-228p+235\right)f_{i+1}^k+\left(25p^2-52p+27\right)f_{i+2}^k\big] \end{aligned} \tag{8}$$

Similarly by substituting $n = 3,4,5,\ldots$, we obtain 6-point relaxed, 6-point, 7-point relaxed subdivision schemes respectively.

3. ANALYSIS OF THE SCHEMES

Aim of this section is to present the analysis of proposed family of binary approximating subdivision schemes. Here we only present the analysis of one family member A_2 of binary subdivision schemes. The analysis of rest of the schemes are similar.

We use Laurent polynomial (symbol) method [18] to calculate integer class continuity, degree of generation and degree of reproduction of the A_n schemes. Moreover, Holder regularity analysis is done by using Riouls [19] method. Using [18], the subdivision scheme with symbol $A_n(z)$ reproduces polynomials of degree d with respect to the parameterizations $\tau = A_n(1)/2$ if and only if

$$A_n^k(-1) = 0, \text{ and } A_n^k(1) = 2\prod_{j=0}^{k-1}(\tau - j), \quad \mathrm{k} = 0,1,2,\ldots,\mathrm{d} \tag{9}$$

Polynomial reproduction of degree d requires polynomial generation of degree d.

Theorem-1: The 5-point binary subdivision scheme A_2 is C^4-continuous.

Proof: We re-write Equation (7) as:

$$A_2(z) = \left(\frac{1+z}{2}\right)^4 b(z)$$

where

$$b(z) = \frac{1}{24z^4}\left[p^2 + \left(21p^2 + 2p\right)z + \left(-22p^2 + 44p + 1\right)z^2 + \left(-22p^2 + 23p\right)z^3 + \left(21p^2 - 44p + 23\right)z^4 + \left(p^2 - 2p + 1\right)z^5\right]$$

Consider a difference scheme S_b corresponding to b(z) and S_c corresponding to $c(z)$ obtained from b(z), such that b(z) = (1+z)c(z) where

$$c(z) = \frac{1}{24z^4}\left[p^2 + \left(20p^2 + 2p\right)z + \left(-42p^2 + 42p + 1\right)z^2 + \left(20p^2 - 42p + 22\right)z^2 + \left(p^2 - 2p + 1\right)z^3\right]$$

To prove C^4-continuity of the scheme A_2, we need to show that S_c is contractive. For contractiveness of S_c

$$A_n^k(-1) = 0, \text{ and } A_n^k(1) = 2\prod_{j=0}^{k-1}(\tau - j), \quad k = 0,1,2,\ldots,d$$

The condition for $\|S_c\|_\infty < 1$ is p∈(0,1). Therefore by [20], if $\|S_c\|_\infty < 1$, then c(z) is contractive and b(z) is convergent. If b(z) is convergent then the scheme Equation (8) is C^4-continuous. Which completes the proof.

Remark-1: The 5-point binary approximating subdivision scheme Equation (8) is C^6-continuous atp = 1/2.

Theorem-2: The lower and upper bounds on the Hlder regularity of the scheme Equation (8) is 6.1256 at p = 1/2.

Proof: From Equation (7), we have $b_0 = 1/12$, $b_1 = 22/12$, $b_2 = 1/12$, k = 7, m = 2 thus 1 = 0,1 and B_0 and $B_1$are the matrices with elements

$(B_0)_{ij} = b_{2+i-2j}$ and $(B_1)_{ij} = b_{3+i-2j}$

where i,j = 1 and 2. Thus B_0 and $B_1$are given by

The eigenvalues of B_0 and B_1 are 1.8333 and 0.08333. The norm infinity of B_0 and B_1 are 1.8333 and 1.8333 respectively. For bounds on Holder regularity, we calculate, where μ is joint spectral radius of the matrices of β_0 and β_1, that is, $\mu = \rho(B_0,B_1)$. Then by [19], lower bound and upper bound on Holder regularity are defined as:

So, max(1.8333,0.08333)≤ μ ≤max(1.8331, 1.8333). So the lower and upper bounds on the Holder regularity is 6.1256. This completes the proof.

Theorem-3: The degree of the polynomial generation of 5-point binary approximating subdivision scheme Equation (8) is 4.

Proof: By Equation (7), we can re-write the Laurent polynomial as:

$A_2(z) = 1+z)^{4+1}\, b_1(z)$

where

Hence by [18], generation degree is 4.

Remark-2: At p = 1/2, the degree of generation of the scheme Equation (8) is 6.

Theorem-4: The 5-point binary approximating subdivision scheme Equation (8) has linear reproduction with parametrization depends on the value of p.

Proof: By taking the first derivative of Equation (7) and put z = 1, we get $A'_2(1) = -2p^2 - 10p + 9$. This implies that $\tau = 1/2\ (-2p^2 - 10p + 9)$, for different values of p, the scheme corresponding to thesymbol A_2 has dual as well as primal parametrization. We can easily verify that first

and second derivative of A_2 at z = - 1 are equal to 0. Further we can also verify Equation (9) for k = 0 and 1. This completes the proof.

In Table 1, we present the complete analysis of some family members of the proposed family of binary approximating subdivision schemes. Here we see that at p = 1/2 the order of continuity and degree of generation have been increased. Moreover, we also present the Holder regularity analysis.

Table 1 shows the analysis of proposed family of schemes, here n, A_n, GD, $GD_{1/2}$, RD, OC, $OC_{1/2}$ and $HR_{1/2}$, represent trial number, proposed schemes, degree of generation, degree of generation at p = 1/2, degree of reproduction, order of continuity, order of continuity at p = 1/2 and Holder regularity of proposed schemes at p = 1/2 respectively.

3.1 Limit Stencil

The limit stencil is a way to obtain a point on the limit curve by using initial control points. The procedure for calculating the limit stencils is presented in [18]. In Table 2, we present the limit stencil of some family members of binary approximating subdivision schemes at p = 1/2.

Table 2 shows the limit stencils of some family members of family of binary approximating subdivision schemes at p = 1/2.

4. CONVEXITY PRESERVATION

In this section, we show the convexity preservation of the scheme A_2. It is clear from Fig. 1(a-c) that if initial control points are strictly convex. Then the limit curves generated by the scheme corresponding toA_2 show positively, normal and negatively skewed behavior on convex data for p< 1/2, p = 1/2 and p> 1/2 respectively.

Theorem-5: Suppose that initial control points $\left\{p_i^0\right\}_{i\in Z}$ are strictly convex, i.e. $p_i^0, \forall\ i \in Z$. Let $d_i^k = 2^{2k+1}\left(f_{i-1}^k - 2f_i^k + f_{i+1}^k\right)$ be defined as second order differences, $r_i^k = \frac{d_{i+1}^k}{d_i^k}, R^k = \max\left\{r_i^k, \frac{1}{r_i^k}\right\}, \forall k \geq 0,\ k \in Z, i \in Z,$ the parameter p satisfies $0 < p < 1$ and $\lambda \in Z$,

where

$$\beta_1 < \lambda < \beta_2 \text{ and } \frac{1}{\lambda} \leq R^0 \leq \lambda,$$

With

$$\beta_1 = \frac{1}{21p^2 - 90p + 70}\left(22p^2 - 44p - 13 + \sqrt{p^4 + 92p^2 - 66p^2 + 1004p + 169}\right)$$

TABLE 2. LIMIT STENCILS ANALYSIS OF FAMILY OF SUBDIVISION SCHEMES

An	Limit Stencils
A_1	{0.0027, 0.0715, 0.2487, 0.2127, 0.1489, 0.0605}
A_2	{0.0021, 0.0840, 0.4139, 0.4139, 0.0840, 0.0021}
A_3	{0.0004, 0.0268, 0.2389, 0.4722, 0.2365, 0.0263, 0.0003}
A_4	{0.0007, 0.0096, 0.1024, 0.3603, 0.3811, 0.1299, 0.0364, 0.0084, 0.0005}

TABLE 1. ANALYSIS OF FAMILY OF SUBDIVISION SCHEMES

n	A_n	GD	RD	OC	$OC_{1/2}$	$GD_{1/2}$	$HR_{1/2}$
1	A_1	4	1	C^4	C^5	5	5.1256
2	A_2	4	1	C^4	C^6	6	6.1256
3	A_3	4	1	C^4	C^7	7	7.1256
4	A_4	4	1	C^4	C^8	8	8.1256
5	A_5	4	1	C^4	C^9	9	9.1256

and

$$\beta_2 = \frac{1}{21p^2 - 90p + 70}\left(22p^2 - 44p - 13 - \sqrt{p^4 + 92p^3 - 66p^2 + 1004p + 169}\right)$$

If

$$d_i^k > 0, \frac{1}{\lambda} \le R^k \le \lambda, \forall k \ge 0, k \in Z, i \in Z \quad (10)$$

then 5-point binary approximating subdivision scheme Equation (8) is convexity preserving.

Proof: Using second order divided difference formula

$$d_i^{k+1} = 2^{2k+1}\left(f_{i-1}^{k+1} - 2f_i^{k+1} + f_{i+1}^{k+1}\right) \quad (11)$$

By using Equation (11), the scheme Equation (8) become

$$d_{2i}^{k+1} = \frac{d_i^k}{48}\left[p^2 d_{i-2}^k + \left(21p^2 + 48p + 1\right)d_{i-1}^{k+1} + \left(-45p^2 + 70\right)d_i^k + \left(23p^2 - 48p + 25\right)d_{i+1}^{k+1}\right] \quad (12)$$

$$d_{2i+1}^{k+1} = \frac{1}{48}\left[\left(23p^2 + 2p\right)d_{i-1}^k + \left(-45p^2 + 90p + 25\right)d_i^k + \left(21p^2 - 90p + 70\right)d_{i+1}^k + \left(p^2 - 2p + 1\right)d_{i+2}^k\right] \quad (13)$$

We use mathematical induction to prove $d_i^k > 0$ and $\frac{1}{\lambda} \le R^k \le \lambda$. When k=0, it is obvious from statement $d_i^0 > 0$ and $\frac{1}{\lambda} \le R^0 \le \lambda$.

$$d_i^0 > 0 \quad \text{and} \quad \frac{1}{\lambda} \le R^0 \le \lambda.$$

Suppose $d_i^k > 0$ and $\frac{1}{\lambda} \le R^k \le \lambda$ is true for*k*. Now we prove for k+1. By re-writing Equation (10) as:

$$d_{2i}^{k+1} = \frac{d_i^k}{48\lambda}\left[\lambda^2\left(21p^2 + 48p + 1\right) + \lambda\left(-44p^2 + 70\right) + \left(23p^2 - 48p + 25\right)\right] \quad (14)$$

$$d_{2i}^{k+1} > 0, \text{ for } \beta_1 < \lambda < \beta_2$$

Where

$$\beta_1 = \frac{1}{21p^2 - 90p + 70}\left(22p^2 - 44p - 13 + \sqrt{p^4 + 92p^3 - 66p^2 + 1004p + 169}\right)$$

and

$$\beta_2 = \frac{1}{21p^2 - 90p + 70}\left(22p^2 - 44p - 13 - \sqrt{p^2 + 92p^2 - 66p^2 + 1004p + 169}\right)$$

Now consider Equation (13) to show $d_{2i+1}^{k+1} > 0$

$$d_{2i+1}^{k+1} = \frac{d_i^k}{48}\left[\left(23p^2 + 2p\right)\frac{d_{i-1}^k}{d_i^k} + \left(-45p^2 + 90p + 25\right)\frac{d_i^k}{d_i^k} + \left(21p^2 - 90p + 70\right)\frac{d_{i+1}^k}{d_i^k} + \left(p^2 - 2p + 1\right)\frac{d_{i+2}^k}{d_i^k}\right]$$

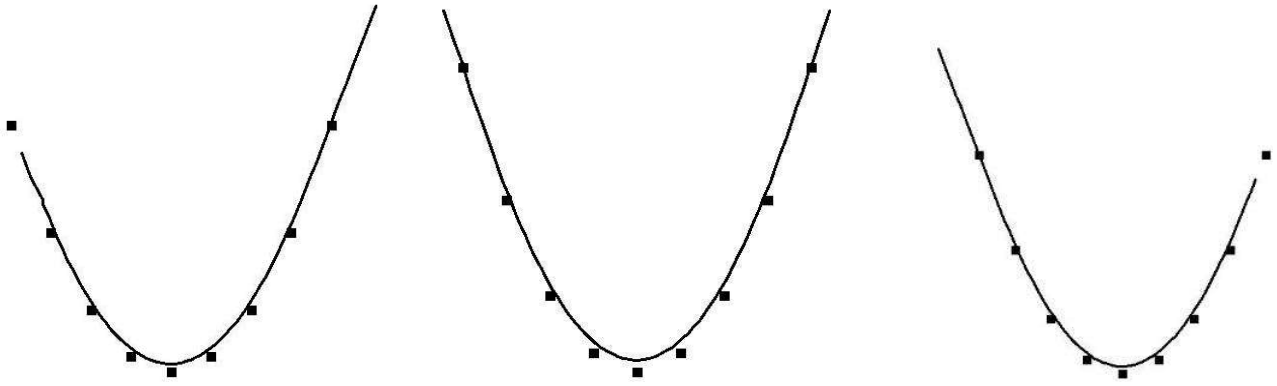
This implies

$$d_{2i+1}^{k+1} - \frac{d_i^k}{48\lambda}\left[\lambda^2\left(21p^2 - 90p + 70\right) + \left(-44p^2 + 88p + 26\right)\lambda + \left(23p^2 + p\right)\right] \quad (15)$$

$$d_{2i+1}^{k+1} > 0, \text{ for } \beta_1 < \lambda < \beta_2 \text{ with } p \in (0,1)$$

Thus

$$d_{2i+1}^{k+1} > 0, \forall k \ge 0, k \in Z, i \in Z$$



(a) Positively Skewed A_2 at p = *1/10*

(b) Normal A_2 at p = 1/2

(c) Negatively Skewed A_2 at p = *1/10*

FIG. 1. PRESENT THE BEHAVIOR OF THE SCHEME A_2

To prove $\frac{1}{\lambda} \le R^{k+1} \le \lambda, \forall k \ge 0$ it is sufficient to prove that

$$\frac{1}{\lambda} < r_i^{k+1} < \lambda, \forall i \in Z, \forall k \ge 0, k \in Z, i \in Z$$

Since we know that $r_{2i}^{k+1} = \frac{d_{2i+1}^{k+1}}{d_{2i}^{k+1}}$

$$r_{2i}^{k+1} - \lambda = \frac{d_{2i+1}^{k+1}}{d_{2i}^{k+1}} - \lambda = \frac{d_{2i+1}^{k+1} - \lambda d_{2i}^{k+1}}{d_{2i}^{k+1}} = \frac{A}{B}$$

where $A = d_{2i+1}^{k+1} - \lambda d_{2i}^{k+1}$ and d_{2i}^{k+1}

From Equation (12), we have already proved that $d_{2i}^{k+1} > 0$ So, $B > 0$ and the numerator satisfiers

$$A = (23p^2 + 2p)\frac{1}{r_{i-1}^k} + (-45p^2 + 90p + 25) + (21p^2 - 90p + 70)r_i^k + (p^2 - 2p + 1)r_i^k r_{i+1}^k$$
$$- \lambda p^2 \frac{1}{r_{i-1}^k}\frac{1}{r_{i-2}^k} - \lambda(21p^2 + 48p + 1)\frac{1}{r_{i-1}^k} - \lambda(-45p^2 + 70) - \lambda(23p - 48p + 45)r_i^k$$

This implies

$$A = \frac{1}{r_{i-1}^k}(23p^2 + 2p - 21p^2\lambda - 48p\lambda - \lambda) + r_i^k(21p^2 - 90p + 70 - 23p^2\lambda + 48p\lambda - 25\lambda)$$
$$+ \frac{1}{r_{i-1}^k}\frac{1}{r_{i-2}^k}(-p^2\lambda) + r_i^k r_{i+1}^k(p^2 - 2p + 1) + (-45p^2 + 90p + 25 + 45p^2\lambda - 70\lambda)$$

This further implies

$$A = \frac{1}{r_{i-1}^k}(23p^2 + 2p - 21p^2\lambda - 48p\lambda - \lambda - p^2) + r_i^k\{21p^2 - 90p + 70 - 23p^2\lambda + 48p\lambda$$
$$- 25\lambda + \frac{1}{\lambda}(p^2 - 2p + 1)\} + (-45p^2 + 90p + 25 + 45p^2\lambda - 70\lambda)$$

Again implies

$$A = \lambda(23p^2 + 2p - 21p^2\lambda - 48p\lambda - \lambda - p^2) + \lambda\{21p^2 - 90p + 70 - 23p^2\lambda + 48p\lambda - 25l + 1/\lambda(p^2 - 2p + 1)\} + (45p^2 + 90p + 25 + 45p^2\lambda - 70\lambda)$$

Furthermore

$$A = \lambda^2(-44p^2 - 26) + \lambda(88p^2 - 88p) + (-44p^2 - 88p + 26) \quad (16)$$

This implies $A < 0$ for $b_1 < l < b_2$. Therefore, $r_{2i}^{k+1} < \lambda, \forall i \in Z$.

Similarly

$$\frac{1}{r_{2i}^{k+1}} - \lambda = \frac{d_{2i}^{k+1}}{d_{2i+1}^{k+1}} - \lambda = \frac{d_{2i}^{k+1} - \lambda d_{2i+1}^{k+1}}{d_{2i+1}^{k+1}} = \frac{C}{D}$$

where $C = d_{2i}^{k+1} - \lambda d_{2i+1}^{k+1}$ and $D = d_{2i+1}^{k+1}$

From Equation (13), we have already proved that $d_{2i+1}^{k+1} > 0$ so, $D > 0$ and the numerator satisfies

$$C = p^2 + (21p^2 + 48p + 1)r_{i-2}^k + (-45p^2 + 70)$$
$$r_{i-2}^k r_{i-1}^k + (23p^2 - 4p + 25)r_{i-2}^k r_{i-1}^k r_i^k - \lambda(23p^2 + p)$$
$$r_{i-2}^k - \lambda(45p^2 + 90p + 25)r_{i-2}^k r_{i-1}^k - \lambda(21p^2 - 90p + 70)$$
$$r_{i-2}^k r_{i-1}^k r_i^k - \lambda(p^2 - 2p + 1)r_{i-2}^k r_{i-1}^k r_i^k r_{i+1}^k$$

This implies

$$C = p^2 + (21p^2 + 48p + 1 - 23p^2\lambda - 2p\lambda)$$
$$r_{i-2}^k + (-45p^2 + 70 + 45p^2\lambda - 90p\lambda - 25\lambda)$$
$$r_{i-2}^k r_{i-1}^k + (23p^2 - 4p + 25 - 21p^2\lambda + 90p\lambda - 70\lambda)$$
$$r_{i-2}^k r_{i-1}^k r_i^k - \lambda(p^2 - 2p + 1)r_{i-2}^k r_{i-1}^k r_i^k r_{i+1}^k$$

This again implies

$$C = p^2 + (21p^2 + 48p + 1 - 23p^2\lambda - 2p\lambda)$$
$$r_{i-2}^k + (-25p^2 + 70 + 45p^2\lambda - 90p\lambda - 25\lambda + 23p^2\lambda - 4p\lambda +$$
$$25\lambda - 21p^2\lambda^2 + 90p\lambda^2 - 70\lambda^2 - p^2\lambda + 2p\lambda - \lambda)r_{i-2}^k r_{i-1}^k$$

Further implies

$$C = p^2 + \lambda(21p^2\lambda + 48p\lambda + 1 - 23p^2\lambda^2 - 2p\lambda^2 - 45p^2 + 70 + 45p^2\lambda - 90p\lambda - 25l + 23p^2\lambda - 4p\lambda + 25l - 21p^2\lambda^2 + 90p\lambda^2 - 70\lambda^2 - p^2\lambda + 2p\lambda - \lambda)$$

Furthermore

$$C = \lambda^2(-44p^2 + 88p - 70) + \lambda(88p^2 - 88p) + (-44p^2 + 70) \quad (17)$$

This implies $C<0$ for $\beta_1<\lambda<\beta_2$.

Thus $\frac{1}{r_{2i}^{k+1}}-\lambda=0.$ Combining Equations (16) and Equation (17), it can be written as:

$$\frac{1}{\lambda}\le r_{2i}^{k+1}\le\lambda,\forall i\in Z$$

Similarly, we can show that

$$\frac{1}{\lambda}\le r_{2i+1}^{k+1}\le\lambda,\ \forall i\in Z$$

Hence $d_i^k>0$ and $\frac{1}{\lambda}\le R^k\le\lambda$ is satisfied for k + 1. Hence proposed scheme A_2 preserves convexity. This completes the proof.

5. COMPARISON AND APPLICATIONS

In this section, we present the comparison and applications of the proposed family of schemes.

5.1 Comparison of Continuity Analysis

Here we present the comparison of continuity analysis of the proposed family of schemes with existing parametric subdivision schemes. It is clear from Table 3 that our proposed family gives higher continuity comparative to the existing parametric subdivision schemes.

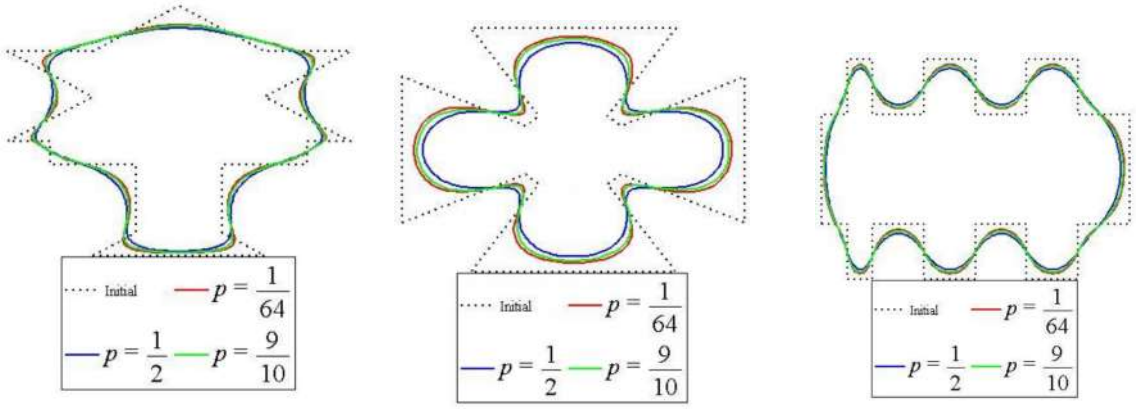
Table 3 shows the comparison of continuity analysis. Here, *E, OC,* A_n and $OC_{1/2}$ denote the existing schemes, order of continuity of existing schemes, proposed family of schemes and continuity of proposed schemes at p = 1/2 respectively.

5.2 Applications

Here we discuss the visual performance of the proposed family of subdivision schemes. The control polygons are drawn by doted lines and the smooth curves obtained by our proposed schemes by full lines. Fig. 2(a-c) represents the applications of proposed scheme A_2 at p = 1/64, 1/2 and 9/10. Fig. 3(a-f) is the applications of scheme A_2 at different values of p.

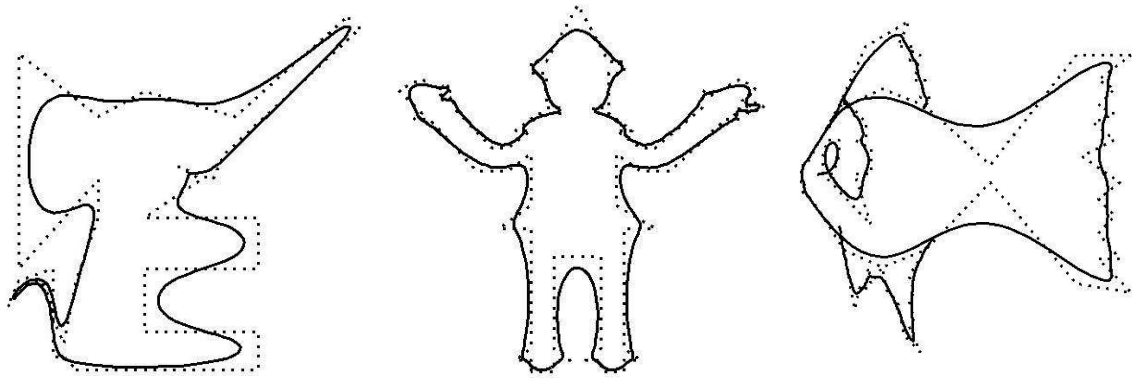
TABLE 3. COMPARISON TABLE

E	OC	A_n	$OC_{1/2}$
5-Point [20]	C^2 if $\frac{1}{324}<w<\frac{1}{162}$	A_2	C^6
5-Point [8]	C^4		
5-Point [13]	C^5		
5-Point [9]	C^2 if $\frac{-3}{64}<w<\frac{1}{162}$		
5-Point [21]	C^4		
6-Point [22]	C^7		
6-Point [10]	C^2 if $w\in[0,0.0425]$	A_4	C^8
6-Point [10]	C^3 if $w\in[0.0139, 0.0143]$		
6-Point [11]	C^6		
6-Point [12]	C^2 if $\frac{7}{972}<w<\frac{11}{1215}$		
6-Point [23]	C^3		
6-Point [24]	C^4		

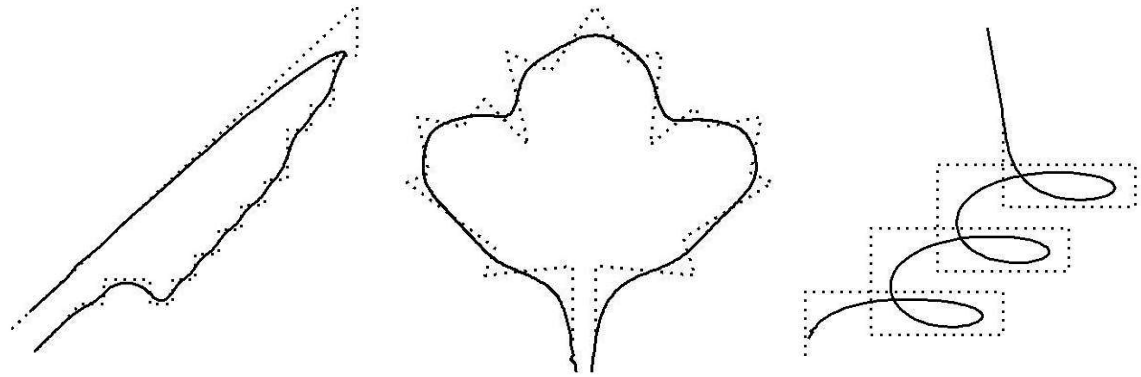


(a) (b) (c)

FIG. 2. PRESENT COMPARISON OF THE LIMIT CURVES FOR CLOSE POLYGONS PRODUCED BY THE SCHEME CORRESPONDING TO A_2



(a) p = 1/10 (b) p = 9/10 (c) p = 1/2



(d) p = 2/9 (e) p = 1/2 (f) p = 1/10

FIG. 3. PRESENT LIMIT CURVES FOR OPEN AND CLOSE POLYGONS PRODUCED BY THE SCHEME CORRESPONDING TO A_2

6. CONCLUSION

In this paper, Binomial probability distribution is used for the construction of new family of binary approximating subdivision schemes. A complete analysis of some family members of the proposed family is presented. We also analyzed that our proposed family give high continuity at $p = 1/2$ comparative to the existing subdivision schemes. At all other values of probability parameter p the results of subdivision schemes remain same.

The parametrization of the subdivision schemes changes at different values of *p*. Subdivision schemes give skewed behavior on the convex data due to the probability parameter. Visual performances of the proposed family of subdivision schemes are also discussed.

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