Exact Solutions on the Oscillating Plate of Maxwell Fluids

KASHIF ALI ABRO*, ASIF ALI SHAIKH**, AND SANAULLAH DEHRAJ***

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ABSTRACT

This work is related to establish the exact solutions of sine hyperbolic and cosine hyperbolic oscillations of Maxwell fluid over the velocity field and shear stress. Under the effects of sine hyperbolic and cosine hyperbolic oscillations, the general solutions are derived for the motions of incompressible Maxwell fluid. For the sack of the general solutions the mathematical techniques of integral transformations (Laplace and Fourier Sine transforms) are applied. We have expressed the obtained solutions under form of theorem of convolutions product and integral notation, satisfying the boundary and initial conditions. The expressions for similar solutions are specialized as a limiting case of Newtonian fluid.

Key Words: Graphical Illustrations, Sine and Cosine Oscillations, Maxwell Fluids, Integral Transforms.

1. INTRODUCTION

The oscillations due to sine and cosine are of great interest in practical as well as theoretical domains and have lot of importance and applications in the scientific and technological field such as oil exploration, chemical engineering, bio-engineering and different industries. Among the comparison of non-Newtonian and Newtonian fluid, the non-Newtonian fluid has diverted much attention among the scientists and researchers as well. On In addition, Navier Stokes equation is inadequate to detect the behavior of non-Newtonian fluid. The complicated fluids do not obey the retardation and relaxation phenomenon rheologically. Under this process several models are presented for determination of fluid flows. These fluids are categorized in three types which are; integral, differential and rate types. The most influential type is the differential type which detects the response of fluids with respect to their slight memory that is applicable in dilute polymeric solutions [1-6]. On the other hand, with reference to polymeric melts the integral model is more enough for detection of behavior of memory phenomenon. The abundance of literature for viscoelastic behavior of fluid lies in non-Newtonian such as second grade and third grade do not detect the phenomenon related to retardation and relaxation. But the models of the types such as second grade, Maxwell, Oldroyd-B and Burger are very popular. The work in this paper is based on the analysis of Maxell fluid which is considered for the observation of the phenomenon like memory effects and elasticity. The numerical simulation to visualize certain experimental data is the main advantage of Maxwell model which is highly applicable in the biological and polymeric liquids[7-20]. The focal communication of this note is to have the solutions for the partial governing differential equations using mathematical transformations techniques which are Fourier sine transforms and Laplace transforms. The velocity field $u(y,t)$ and shear stress $\tau(y,t)$ are found out by satisfying all the boundary and initial conditions. The

* Lecturer, Department of Basic Sciences and Related Studies, Mehran University of Engineering & Technology, Jamshoro.
** Assistant Professor, Department of Basic Sciences and Related Studies, Mehran University of Engineering & Technology, Jamshoro.
*** Lecturer, Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science, & Technology, Nawabshah.

obtained solutions are written as the limiting case of analysis. Finally several graphical discussions and illustrations are considered depending over the variations of parameters. As for the checking of exactness and accuracy, we also showed that for small values of the rheological parameters λ only, the diagrams of the solutions are very nearly identical to those corresponding to the known solutions for Newtonian fluids.

2. GOVERNING EQUATIONS

The stress tensor for Cauchy under the consideration of Maxwell fluid is detected as [19-20]:

$$S - pI = T, \lambda \frac{\delta S}{\delta t} + S = A_1 \mu \tag{1}$$

$$\frac{dS}{dt} - LS - SL^T - S - LS - SL^T = \frac{\delta S}{\delta t} \tag{2}$$

is the upper convected derivative. We assume a an extra-stress tensor **S** and velocity field **V o**f the form

$$\boldsymbol{V} = \boldsymbol{V}(y,t) = u(y,t)\boldsymbol{i}, \boldsymbol{S}(y,t) = \boldsymbol{S} \tag{3}$$

At the moment $t = 0$, the fluid is at rest then

$$\boldsymbol{V} = (y,0) = 0, \boldsymbol{S} = (y,0) = 0 \tag{4}$$

From equations (1) and (4) imply $S_{yz} = S_{yy} = S_{zz} = S_{xz} = 0$ and

$$\left(\tau + \lambda \frac{\partial \tau}{\partial t}\right) = \mu \frac{\partial u}{\partial y} \tag{5}$$

where the non-zero shear stresses are $S_{xy} = \tau$. The balance of linear momentum reduces to:

$$\rho u_t = \tau_t - p_x \tag{6}$$

In the absence of body forces, eliminating τ between Equations (5) and (6), we have the partial differential equations under the form:

$$\left(\frac{\partial u}{\partial t} + \lambda \frac{\partial^2 u}{\partial t^2}\right) = \mu \frac{\partial^2 u}{\partial^2 t} - \frac{1}{\rho}\left(1 + \lambda \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x} \tag{7}$$

Where y,t > 0 and the kinematic viscosity of the fluid is $v = \mu/\rho$, then equations govern the Maxwell fluid are:

$$v \frac{\partial^2 u}{\partial^2 t} = \left(\frac{\partial u}{\partial t} + \lambda \frac{\partial^2 u}{\partial t^2}\right) \tag{8}$$

$$\mu \frac{\partial u}{\partial y} = \left(\tau + \lambda \frac{\partial \tau}{\partial t}\right) \tag{9}$$

The concerned problem with initial and boundary conditions is:

$$\tau(y,0) = 0, u(y,0) = \frac{\partial u(y,0)}{\partial t} = 0 \tag{10}$$

$$u(0,t) = UH(t) \sinh\omega t \text{ or } UH(t) \cosh\omega t \; t \geq 0 \tag{11}$$

Moreover, the considerable conditions with Heaviside function H(t).

$$u(y,t) \frac{\partial u(y,t)}{\partial t} \to 0 \text{ as } y > 0, y \to \infty \tag{12}$$

have to be also satisfied.

3. CALCULATION OF PROBLEM

3.1 Velocity Components for Sine Oscillation

For sacking the solutions, we shall use the Fourier sine transforms [21].

$$\left\{\frac{\partial u_s}{\partial t} + \lambda \frac{\partial^2 u_s}{\partial^2 t}\right\} = vH\xi \sqrt{\frac{2}{\pi}} H(t) \sinh \omega t - \xi^2 v u_s \tag{13}$$

Where u_s is Fourier sine transform and has to satisfy the initial conditions:

$$u_s(\xi,0) = u_{st}(\xi,0) = 0, \xi > 0 \tag{14}$$

By applying the Laplace transform [22] to equation (13) and having in mind the initial conditions from Equation (10), we find that:

$$\bar{u}_s = \frac{\sqrt{2}\nu U\omega\xi}{\sqrt{\pi}\left(q^2-\omega^2\right)\left(\xi^2\nu+q+q^2\lambda\right)} \tag{15}$$

Now, rewriting Equation (15) in very appropriate representation:

$$\bar{u}_s = \left[\frac{\omega U\sqrt{2/\pi}}{\xi\left(q^2-\omega^2\right)} - \frac{\omega U\sqrt{2/\pi}q\left(1+\lambda q\right)}{\left(q^2-\omega^2\right)} \times \frac{1}{\left(\xi^2\nu+q+q^2\lambda\right)}\right] \tag{16}$$

Applying the inverse Fourier sine formula on Equation (16) as:

$$\bar{u} = \left[\frac{2\omega U}{\pi\left(q^2-\omega^2\right)}\int_0^\infty \frac{\sin\left(y\xi\right)}{\xi}d\xi - \frac{2\omega Uq}{\pi\left(q^2-\omega^2\right)} \times \int_0^\infty \frac{\sin\left(y\xi\right)}{\xi}\left(\frac{\lambda q+1}{\xi^2\nu+q+q^2\lambda}\right)d\xi\right] \tag{17}$$

Finally for velocity field, we apply the inverse Laplace transform to Equation (17), the simple expression is:

$$u_s = UH(t)\sinh \omega t - \frac{2\omega UH(t)}{(q_1-q_2)\lambda\pi} \times \int_0^t \int_0^\infty \frac{\sin(y\xi)}{\xi}\cosh \omega-(t-u)\left\{e^{q_1 u}\left(q_1\lambda+1\right)-e^{q_2 u}\left(q_2\lambda+1\right)dud\xi\right\} \tag{18}$$

Where

$$q_2, q_1 = -\frac{\left(1\right)\pm\sqrt{-4\lambda(\nu\xi^2)+(1)^2}}{2\lambda} \tag{19}$$

are the roots of $\xi^2\nu + q + q^2\lambda = 0$ quadratic equation.

3.2 Evaluation of Shear Stress for Sine Oscillations

Apply Laplace transform to Equation (9) to have solution of shear stress, here we determine as:

$$\bar{\tau} = \frac{\mu\bar{u}_y}{(q\lambda+1)} \tag{20}$$

By finding the partial derivative of Equation (17) with respect to y, we get:

$$\bar{u}_y = \left[\frac{2\omega U}{\pi\left(q^2-\omega^2\right)}\int_0^\infty \cos(y\xi)d\xi - \frac{2\omega Uq}{\left(q^2-\omega^2\right)} \times \int_0^\infty \cos(y\xi)\frac{\left(\lambda q+1\right)}{\xi^2\nu+q+q^2\lambda}d\xi\right] \tag{21}$$

Solving Equations (21) in Equation (20):

$$\bar{\tau} = \frac{\mu}{\left(1+\lambda q\right)}\left[\frac{2\omega U}{\pi\left(q^2-\omega^2\right)}\int_0^\infty \cos(y\xi)d\xi - \int_0^\infty \cos(y\xi)\frac{2\omega Uq}{\left(q^2-\omega^2\right)}\frac{\left(\lambda q+1\right)}{\xi^2\nu+q+q^2\lambda}d\xi\right] \tag{22}$$

Simplifying Equation (22):

$$\bar{\tau} = \frac{\mu}{\left(1+\lambda q\right)}\left[\frac{2\omega U}{\pi\left(q^2-\omega^2\right)}\int_0^\infty \cos(y\xi)d\xi - \int_0^\infty \cos(y\xi)\frac{2\omega Uq}{\left(q^2-\omega^2\right)}\frac{\mu}{\left(1+q\lambda\right)} \times \frac{\left(\lambda q+1\right)}{\xi^2\nu+q+q^2\lambda}d\xi\right] \tag{23}$$

Applying inverse Laplace transform in Equation (23) we get:

$$\tau_s = \frac{2\mu\omega UH(t)}{\left(q_1-q_2\right)\lambda\pi}\int_0^t\int_0^\infty \cos(y\xi) \times \left(e^{q_1 u}-e^{q_1 u}\right)\cosh \omega(t-u)dud\xi \tag{24}$$

Is sought out.

3.3 Velocity Field and Shear Stress for Cosine Oscillations

Under considering the similar procedure of mathematical transformation techniques, the solutions for velocity field and shear stress for cosine oscillations are:

$$u_c = UH(t)\cosh \omega t - \frac{2\omega UH(t)}{\left(q_1-q_2\right)\lambda\pi} \times \int_0^t\int_0^\infty \frac{\sin(y\xi)}{\xi}\sinh \omega(t-u)\left\{e^{q1u}\left(q_1\lambda+1\right)-e^{q2u}\left(q_2\lambda+1\right)\right\}dud\xi \tag{25}$$

$$\tau_c = -\frac{2\mu\omega UH(t)}{\left(q_1-q_2\right)\lambda\pi}\int_0^t\int_0^\infty \cos(y\xi) \times \left(e^{q1u}-e^{q1u}\right)\sinh \omega(t-u)dud\xi \tag{26}$$

are obtained respectively.

4. SPECIAL CASES

4.1 Newtonian Fluid

Solving Equations (18, 24, 25 and 26) for $l \rightarrow 0$ as the limit and applying following possible mathematical facts:

$$\lim_{\lambda\to 0}\left(q_1-q_2\right) = \lambda = 1, \lim_{\lambda\to 0} q_2 = -\infty, \lim_{\lambda\to 0} q_1 = -\xi^2\nu$$

$$u_{SN} = UH(t)\sinh \omega t - \frac{2\omega UH(t)}{\pi} \times \int_0^t\int_0^\infty \frac{\sin(y\xi)}{\xi}\cosh \omega(t-u) \times \exp\left(-\xi^2\nu\right)udud\xi \tag{27}$$

$$\tau_{SN} = \frac{2\mu\omega UH(t)}{\pi}\int_0^t\int_0^\infty \cos(y\xi) \times \cosh \omega(t-u)\exp(-\xi^2\nu)udud\xi \tag{28}$$

$$u_{CN} = UH(t)\cosh \omega t - \frac{2\omega UH(t)}{\pi} \times \int_0^t\int_0^\infty \frac{\sin(y\xi)}{\xi}\sinh \omega(t-u) \times \exp(-\xi^2\nu)udud\xi \tag{29}$$

$$\tau_{CN} = -\frac{2\mu\omega UH(t)}{\pi}\int_0^t\int_0^\infty \cos(y\xi) \times \sinh \omega(t-u)\exp(-\xi^2\nu)udud\xi \tag{30}$$

are achieved.

5. COMPUTATIONAL RESULTS AND DISCUSSION

In this paragraph, we have developed exact solutions for the effects and oscillations of sine hyperbolic and cosine hyperbolic for Maxwell fluid. To depict and capture certain relevant physical aspects, we have drawn several graphs related to motion of Maxwell fluid under effects and oscillations of sine and cosine. The graphical as well as numerical profiles of velocity field and shear stress are illustrated for incompressible Maxwell fluid with respect to emerging parameters of variation of interest. From the general solutions, the similar solutions are particularized as a Newtonian solution also the comparison and contrast has been depicted for both types of solutions graphically and numerically. Fig. 1 is arranged for the oscillations of sine and cosine for Maxwell fluid, where the variations of t display the effects of increasing function of velocity field and shear stress. Fig. 2 provides the increasing motion with regard to the increasing value of v. Fig. 3 gives interesting results under the effects of relaxation time λ to have intersecting motion of fluid. Fig. 4 is depicted for amplitude in which both the velocity field as well as the shear stress is increasing function of for the motion of incompressible Maxwell fluid.

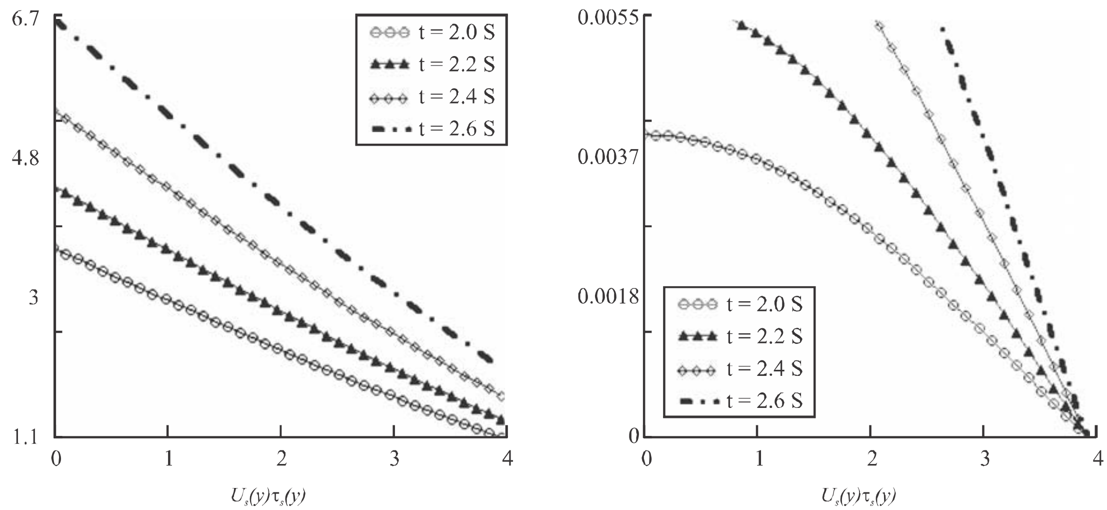

FIG. 1. PLOT OF THE VELOCITY FIELD AND THE SHEAR STRESSF ROM EQUATIONS (18) AND (24), FOR M,N,U,Λ, AND Ω WITH VALUES 1.52, 0.63,2, 2, AND 2 FOR VARIOUS VALUES OF T

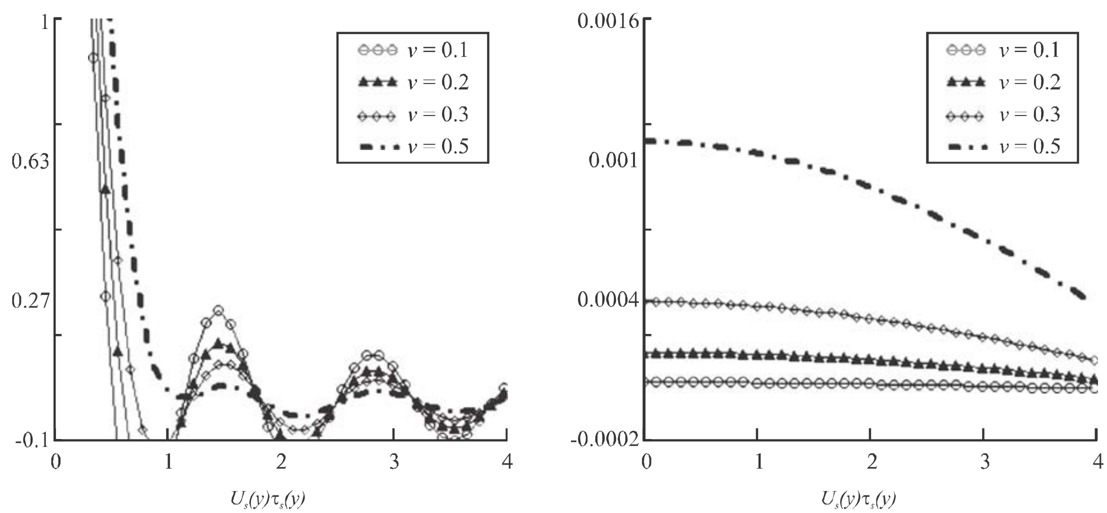

FIG. 2. PLOT OF THE VELOCITY FIELD AND THE SHEAR STRESSF ROM EQUATIONS (18) AND (24), FOR M,P,U,Λ, AND T WITH VALUES 1.52, 2.413,1, 2, AND 2 S RESPECTIVELY FOR VARIOUS VALUES OF N

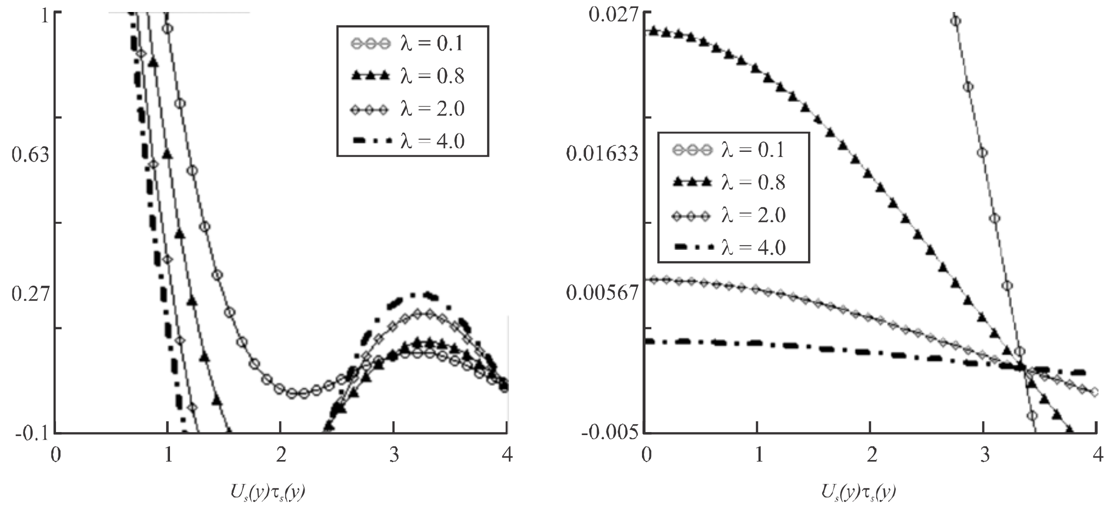

FIG. 3. PLOT OF THE VELOCITY FIELD AND THE SHEAR STRESS FROM EQUATIONS(18) AND (24), FOR M,N,U,Ω, AND WITH VALUES 1.52, 0.63,1,2, AND 2 RESPECTIVELY FORVARIOUS VALUES OF Λ

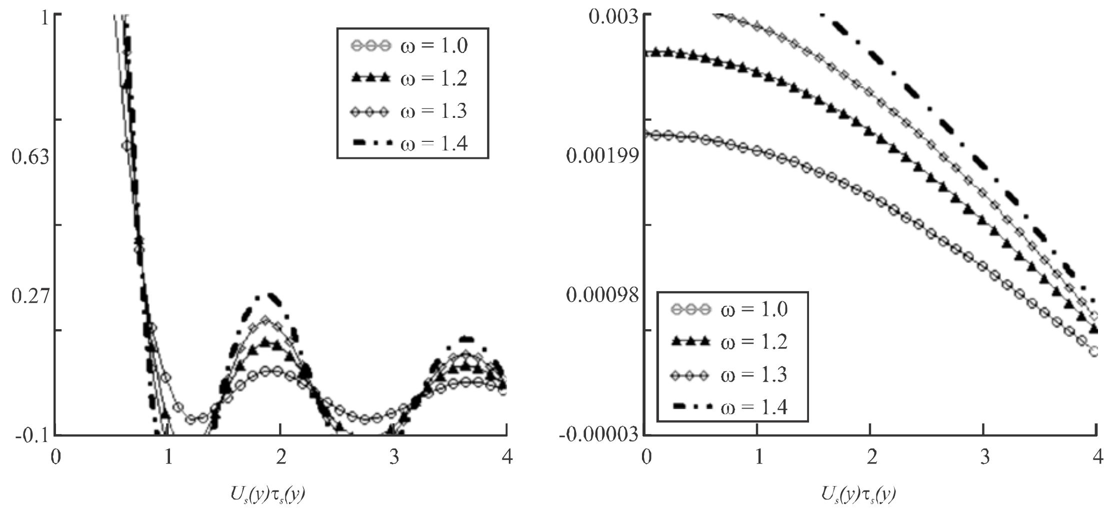

FIG. 4. PLOT OF THE VELOCITY FIELD AND THE SHEAR STRESS FROM EQUATIONS(18) AND (24), FOR M,N,U,Λ, AND WITH VALUES 1.52, 0.63,1, 2, AND 2 RESPECTIVELY FOR VARIOUS VALUES OF Ω

6. CONCLUSION

The major findings related to this paper are exact solutions over hyperbolical oscillations of sine and cosine for Maxwell fluid. The main focus is given to use mathematical transformation techniques for the solutions of such type of Maxwell fluid problems arising in nature. Different types of conditions are also imposed on the general solutions to verify the results. Further it is also analyzed that as the time increases the motion of the fluid as well as shear stresses increases and also as viscosity increases the profile of velocity field is oscillating and shear stresses are increasing. In addition, as the amplitude of fluid increases velocity field oscillates and shear stresses increases.

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