Abel-Grassmann's Groupoids of Modulo Matrices

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ABSTRACT

The binary operation of usual addition is associative in all matrices over R. However, a binary operation of addition in matrices over Z_n of a nonassociative structures of AG-groupoids and AG-groups are defined and investigated here. It is shown that both these structures exist for every integer $n \geq 3$. Various properties of these structures are explored like: (i) Every AG-groupoid of matrices over Z_n is transitively commutative AG-groupoid and is a cancellative AG-groupoid if n is prime. (ii) Every AG-groupoid of matrices over Z_n of Type-II is a T^3-AG-groupoid. (iii) An AG-groupoid of matrices over Z_n; $G_{nAG}(t,u)$, is an AG-band, if $t+u=1 \pmod n$.

Key Words: AG-groupoid and AG-group of Matrices over Z_n, T^3-AG-groupoid, Transitively Commutative AG-groupoid, Cancellative AG-groupoid.

1. INTRODUCTION

A magma that satisfies the left invertive law: $(ab)c=(cb)a$ is called left almost semigroup abbreviated as LA-semigroup [1], or Abel-Grassmann's groupoid abbreviated as AG-groupoid [2]. An AG-group G is an AG-groupoid which has the left identity and has inverse of each of its elements. Both these structures are nonassociative in general, and so one has to play the game of brackets in a defined way. In an AG-groupoid G an element $a \in G$ is called idempotent if $a=a^2$ and G is called idempotent or AG-2-band or simply AG-band if each of its elements is idempotent [3]. AG-groupoids generalize commutative semigroups, while an AG-group generaliz an Abelian group. These structures have a variety of applications in geometry, flocks theory, topology, finite mathematics and many more [4-7]. Many new classes of AG-groupoids have recently been introduced and characterized [8-13]. Fuzzification of AG-groupoids and AG-groups has also been done see for instance [14,15]. Shah studied a lot about the AG-groupoids and AG-groups exclusively [4]. However, the construction of these structures, as well as of other algebraic structures remains a difficult job for the researchers. In this article we try to introduce construction of these structures especially in matrices. A matrix A over a field F is a rectangular array of scalars represented by:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}$$

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The rows of matrix A are the p horizontal entries $(a_{11},a_{12},\ldots,a_{1q})$, $(a_{21},a_{22},\ldots a_{2q})$, …, $(a_{p1}, a_{p2},\ldots,a_{pq})$, and the column of A are the q vertical entries:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{p1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{p2} \end{bmatrix} \text{ and } \begin{bmatrix} a_{1q} \\ a_{2q} \\ \vdots \\ a_{pq} \end{bmatrix}$$

Note that a_{ij} represents the entry in i^{th} row and j^{th} column. A matrix with p rows and q columns is called pxq matrix. A matrix having only one row is called row matrix or row vector, and a matrix having only one column is called a column matrix or column vector.

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices of the same order, say pxq. Then $A+B$, is a matrix obtained by adding the corresponding entries from A and B i.e. $A+B=[a_{ij}+b_{ij}]$. The product of the matrix A by a scalar c, cA is a matrix obtained by multiplying each entry of A by c i.e. $cA=[ca_{ij}]$.

Vasantha, et., al. [16] has defined some nonassociative construction for groupoids on matrices as follows:

Definition-1. Let $G_1=\{(x_1,x_2,\ldots,x_q) \mid x_1 \in Z_n; 1 \le i \le q\}; n \ge 3$ be the collection of row matrices over Z_n. Define a binary operation '*' on G_1 as follows:

$$(x_1,x_2,\ldots,x_q)*(y_1,y_2,\ldots,y_q) \cong \left[t(x_1,x_2,\ldots,x_q)+u(y_1,y_2,\ldots,y_q)\right](\text{mod } n)$$
$$\cong \left[(tx_1+uy_1)(\text{mod } n),\ldots,(tx_q+uy_q)(\text{mod } n)\right]$$

Where $t,u \in Z_n-\{0\}$, $t \ne u$ and $(t,u)=1$ for all $(x_1,x_2,\ldots,x_q)$, $(y_1,y_2,\ldots,y_q) \in G_1$.

Thus $(G_1(t,u),*)$ is a groupoid of row matrix over Z_n.

Definition-2. Let $G_2=\{(x_1,x_2,\ldots,x_p)^t \mid x_i \in Z_n; 1 < i \le p\}; n \ge 3$ be the collection of px1 column matrices over Z_n. Choose $t,u \in Z_n-\{0\}$, $t \ne u$ and $(t,u)=1$ for all $(x_1,x_2,\ldots,x_p)^t$, $(y_1,y_2,\ldots,y_p)^t \in G_2$. Define '*' on G_2 as follows:

$$(x_1,x_2,\ldots,x_p)^t*(y_1,y_2,\ldots,y_p)^t = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \cong \begin{bmatrix} (tx_1+uy_1)(mod\ n) \\ (tx_2+uy_2)(mod\ n) \\ \vdots \\ (tx_p+uy_p)(mod\ n) \end{bmatrix}$$

Thus $(G_2(t,u),*)$ is a groupoid of column matrix over Z_n.

Definition-3. Let $G=\{[m_{ij}] \mid 1 \le i \le p, 1 \le j \le q\}$, be the collection of pxq matrices over Z_n; $n \ge 3$. For non zero distinct integers t,u in Z_n, define '*' on G, for two matrices $M=[m_{ij}]$ and $N=[n_{ij}]$ in G as follows:

$$M*N \begin{bmatrix} m_{ij} \end{bmatrix} * \begin{bmatrix} n_{ij} \end{bmatrix} \cong \left[\left(tm_{ij}+un_{ij}\right)(\text{mod } n)\right]$$

Thus $(G(t,u),*)$ is any groupoid on pxq matrix over Z_n. Moreover, we generally represent this groupoid of (row matrix, column matrix or pxq matrix) over Z_n by $G_n(t,u)$. We will use $G(n)$ to denote the class of $G_n(t,u)$ for distinct integers $t,u \in Z_n-\{0\}$, and $(t,u)=1$, that is:

$G(n)=\{G_n(t,u)$, for distinct integers $t,u \in Z_n-\{0\}$, and $(t,u)=1\}$.

By putting some additional conditions on t and u in $G(n)$ we get some other classes of groupoids of matrices over Z_n as follow:

(i) Type-I, if for non-zero distinct integers t,u in Z_n and $(t,u) \ne 1$.

(ii) Type-II, if $t,u \in Z_n-\{0\}$, such that $t=u$.

(iii) Type-III, if $t,u \in Z_n$, where t or u is zero.

In the following section we show the existence of AG-groupoid of matrices over Z_n, and find its relations with some of the already known classes of AG-groupoids.

2. EXISTENCE OF AG-GROUPOID OF MATRICES OVER Z_n

The following theorem shows the existence of AG-groupoid of matrices over Z_n where $n \ge 3$, and indeed

it introduces a simple way of construction of these AG-groupoids of any finite order.

Theorem-1. $G_n(t,u)$, is an AG-groupoid of matrices over Z_n, if $t^2 \cong u(\text{mod } n)$ for any $t,u \in Z_n$.

Proof. Let $G_n(t,u)$ satisfies $t^2 \cong u(\text{mod } n)$ for any $t,u \in Z_n$. To show that $G_n(t,u)$ is an AG-groupoid of row matrices (column matrices or pxq matrices), it is sufficient if we show the left invertive law: $(A*B)*C = (C*B)*A$; $\forall A,B,C \in G_n(t,u)$, holds;

$$(A*B)*C = \left(\left[a_{ij}\right]*\left[b_{ij}\right]\right)*\left[c_{ij}\right] \cong \left[\left(ta_{ij}+ub_{ij}\right)(mod\ \ n)\right]*\left[c_{ij}\right]$$

This implies that,

$$\Rightarrow (A*B)*C \cong \left(t^2\, a_{ij}+tub_{ij}+uc_{ij}\right)(mod\ \ n) \quad (1)$$

and

$$(C*B)*A = \left(\left[c_{ij}\right]*\left[b_{ij}\right]\right)*\left[a_{ij}\right] \cong \left(t^2\, c_{ij}+tub_{ij}+ua_{ij}\right)(mod\ \ n) \quad (2)$$

also

$$A*(B*C) = \left[a_{ij}\right]*\left(\left[b_{ij}\right]*\left[c_{ij}\right]\right) \cong \left(ta_{ij}+utb_{ij}+u^2 c_{ij}\right)(mod\ \ n) \quad (3)$$

This implies that $G_n(t,u)$ is nonassociative AG-groupoid of matrices over Z_n by Equations (1-3).

We denote this AG-groupoid of matrices over Z_n by $G_{nAG}(t,u)$, and $G_{AG}(n)$ will represent the class that contains all AG-groupoids of matrices over Z_n. Now by varying values of t and u and by imposing some additional conditions on t and u, we get different classes of AG-groupoid of matrices over Z_n for some fixed integer $n \geq 3$. Thus the so obtained new classes of AG-groupoids of matrices over Z_n will be denoted by $G_{AG\text{-}I}(n)$, $G_{AG\text{-}II}(n)$ and $G_{AG\text{-}III}(n)$. The following example shows the existence of these AG-groupoids of matrices over Z_n.

Example-1. $G_3(2,1)$ is an AG-groupoid of matrices over Z_3, that is $G_{3AG}(2,1) \in G_{AG}(3)$.

Solution. As $G_3(2,1) \in G(3)$, to show that it is an AG-groupoid of matrices over Z_3, that is, $G_{3AG}(2,1) \in G_{AG}(3)$, we show that it satisfies the left invertive law:

$$(A*B)*C = \left(\left[a_{ij}\right]*\left[b_{ij}\right]\right)*\left[c_{ij}\right] \cong \left[\left(2a_{ij}+b_{ij}\right)(mod\ \ n)\right]*\left[c_{ij}\right]$$

This implies that,

$$\Rightarrow *C \cong \left[\left(a_{ij}+2b_{ij}+c_{ij}\right)(mod\ \ n)\right] \quad (4)$$

and

$$(C*B)*A = \left(\left[c_{ij}\right]*\left[b_{ij}\right]\right)*\left[a_{ij}\right] \cong \left[\left(c_{ij}+2b_{ij}+a_{ij}\right)(mod\ \ n)\right] \quad (5)$$

also

$$A*(B*C) = \left[a_{ij}\right]*\left(\left[b_{ij}\right]*\left[c_{ij}\right]\right) \cong \left[\left(2a_{ij}+2b_{ij}+c_{ij}\right)(mod\ \ n)\right] \quad (6)$$

This implies that $G_{3AG}(2,1) \in G_{AG}(3)$ by Equations (4 and 5), and is nonassociative by Equations (4 and 6).

Example-2. $G_8(6,4)$ is an AG-groupoid of matrices over Z_8 of Type-I, that is $G_{8AG}(6,4) \in G_{AG\text{-}I}(8)$.

The following examples show some various types of AG-groupoids of matrices over Z_n for $n \geq 3$.

Example-3. $G_{AG}(3) = \{G_{3AG}(2,1)\}$ and $G_{AG\text{-}II}(3) = \{G_{3AG}(1,1)\}$.

Example-4. $G_{AG}(4) = \{G_{4AG}(3,1), G_{AG\text{-}II}(4) = \{G_{4AG}(1,1)\}$ and $G_{AG\text{-}III}(4) = \{G_{4AG}(2,0)\}$.

Example-5. $G_{AG}(5) = \{G_{5AG}(4,1), G_{5AG}(3,4)\}$ $G_{AG\text{-}I}(5) = \{G_{5AG}(2,4\}$ and $G_{AG\text{-}II}(5) = \{G_{5AG}(1,1)\}$.

Example-6. $G_{AG}(6) = \{G_{6AG}(5,1)\}$, $G_{AG\text{-}I}(6) = \{G_{6AG}(2,4)\}$ and $G_{AG\text{-}II}(6) = \{G_{6AG}(1,1), G_{6AG}(3,3), G_{6AG}(4,4)\}$ and so on.

The following corollaries are straight away by Theorem-1.

Corollary-1. Any $G_{nAG}(t,u) \in G_{AG\text{-}II}(n)$ is an abelian group (an AG-group), if $t=u=1$.

Corollary-2. $G_{nAG}(t,u) \in G_{AG\text{-}II}(n)$ is a commutative semigroup, if $t=u \neq 1$.

Example-7. AG-groupoids of matrices $G_{6AG}(3,3)$ and $G_{6AG}(4,4)$ of Z_6 are commutative semigroups in $G_{AG\text{-}II}(6)$.

3. SOME PROPERTIES OF AN AG-GROUPOID OF MATRICES OVER Z_n

In this section we investigate the relations of these new classes of AG-groupoid of matrices over Z_n with some of the already known classes of AG-groupoids that includes the following:

(i) T^3-AG-groupoid, if it is;

(a) T^3_l-AG-groupoid, that is, if $a*b=a*c \Rightarrow b*a=c*a$ [17].

(b) T^3_r-AG-groupoid, that is, if $b*a=c*a \Rightarrow a*b=a*c$ [17].

(ii) Transitively commutative AG-groupoid, if $a*b=b*a$ and $b*c=c*b \Rightarrow a*c=c*a$ [17].

(iii) Cancellative AG-groupoid, an element a in an AG-groupoid G is left (right) cancellative, if $a.x=x.y \Rightarrow x=y (x.a=y.a \Rightarrow x=y)$, and G is left (right) cancellative if each of its elements is left (right) cancellative [18].

(iv) AG-band, if $a*a=a$ [3].

Theorem-2. Every AG-groupoid of matrices over Z_n of Type-II that is $G_{AG\text{-}II}(n)$ is a T^3-AG-groupoid.

Proof. To show that $G_{AG\text{-}II}(n)$ is a T^3-AG-groupoid, it is sufficient if we show that an arbitrary AG-groupoid of matrices over Z_n of Type-II is T^3_l-AG-groupoid and T^3_r-AG-groupoid.

For T^3_l-AG-groupoid, let $A,B,C \in G_{AG\text{-}II}(n)$, and

$$A*B = A*C \Rightarrow \left[a_{ij}\right]*\left[b_{ij}\right] = \left[a_{ij}\right]*\left[c_{ij}\right]$$
$$\Rightarrow \left[\left(ta_{ij}+ub_{ij}\right)(mod\ n)\right] \cong \left[\left(ta_{ij}+uc_{ij}\right)(mod\ n)\right]$$

This implies that,

$$\Rightarrow \left[\left(ub_{ij}\right)(mod\ n)\right] \cong \left[\left(uc_{ij}\right)(mod\ n)\right] \quad (7)$$

As $t,u \in G_{AG\text{-}II}(n)$, therefore, by putting $u=t$ in Equation (7) we get:

$$\left[\left(tb_{ij}\right)(mod\ n)\right] \cong \left[\left(tc_{ij}\right)(mod\ n)\right] \quad (8)$$

Now by Equation (8) we get,

$$B*A = \left[b_{ij}\right]*\left[a_{ij}\right] \cong \left[\left(tb_{ij}+ua_{ij}\right)(mod\ n)\right]$$
$$\cong \left[\left(tc_{ij}+ua_{ij}\right)(mod\ n)\right] = \left[c_{ij}\right]*\left[a_{ij}\right] \Rightarrow B*A = C*A$$

by Equations (7)

$$\Rightarrow B*A = \left[c_{ij}\right]*\left[a_{ij}\right] \Rightarrow B*A = C*A$$

Hence $G_{AG\text{-}II}(n)$ is T^3_l-AG-groupoid. Similarly we can show that $G_{AG\text{-}II}(n)$ is T^3_r-AG-groupoid. Hence $G_{AG\text{-}II}(n)$ is T^3-AG-groupoid.

Theorem-3. Every AG-groupoid of matrics over Z_p that is $G_{pAG}(t,u)$ is a T^3-AG-groupoid, if p is prime and $u \in Z_p - \{0\}$.

Proof. To show that every $G_{pAG}(t,u)$ is a T^3-AG-groupoid for $u \in Z_p - \{0\}$ it is sufficient to show that, $G_{pAG}(t,u)$ is T^3_l-AG-groupoid and T^3_r-AG-groupoid.

For T^3_l-AG-groupoid, let $A,B,C \in G_{AG\text{-}II}(n)$, and

$$A*B = A*C \Rightarrow \left[a_{ij}\right]*\left[b_{ij}\right] = \left[a_{ij}\right]*\left[c_{ij}\right]$$
$$\Rightarrow \left[\left(ta_{ij} + ub_{ij}\right)(mod\ p)\right] \cong \left[\left(ta_{ij} + uc_{ij}\right)(mod\ p)\right]$$
$$\Rightarrow \left[u\left(b_{ij} - c_{ij}\right)(mod\ p)\right] \cong [0(mod\ p)]$$

as u is not devisible by p, because a non-zero u and $(b_{ij}\text{-}c_{ij})$ both are less then p, where p is prime. Therefore, $p \mid [b_{ij}\text{-}c_{ij}] \Rightarrow [b_{ij}(\text{mod } p)] \cong [c_{ij}(\text{mod } p)]$, consequently;

$$B*A = \left[b_{ij}\right]*\left[a_{ij}\right] \cong \left[\left(tb_{ij} + ua_{ij}\right)(mod\ p)\right]$$
$$\cong \left[\left(tc_{ij} + ua_{ij}\right)(mod\ p)\right] = \left[c_{ij}\right]*\left[a_{ij}\right] \Rightarrow B*A = C*A.$$

Hence every AG-groupoid of matrices over Z_p is a T^3_l-AG-groupoid. Similarly, we can show that every AG-groupoid of matrices over Z_p is a T^3_r-AG-groupoid. Hence for non zero $u \in Z_p$, every AG-groupoid of matrices over Z_p is a T^3-AG-groupoid.

Example-8. $G_{5AG}(3,4)$ is a T^3-AG-groupoid. However, the result is not true in general.

For example; $G_{8AG}(6,4)$ is not a T^3-AG-groupoid.

From the following theorem it is clear that the collection of AG-groupoids of matrices over Z_n in any class is a subclass of transitively commutative AG-groupoid.

Theorem-4. Every $G_{nAG}(t,u)$ is a transitively commutative AG-groupoid.

Proof. To show that every $G_{nAG}(t,u)$ is a transitively commutative AG-groupoid, we show that an arbitrary AG-groupoid of matrices is transitively commutative AG-groupoid. Let $A,B,C \in G_{nAG}(t,u)$, and

$$A*B = B*A \Rightarrow \left[a_{ij}\right]*\left[b_{ij}\right] = \left[b_{ij}\right]*\left[a_{ij}\right]$$
$$\Rightarrow \left[\left(ta_{ij} + ub_{ij}\right)(mod\ n)\right] \cong \left[\left(tb_{ij} + ua_{ij}\right)(mod\ n)\right]$$

This implies that,

$$\left[\left(t\left(a_{ij} - b_{ij}\right) + u\left(b_{ij} - a_{ij}\right)\right)(mod\ n)\right] \cong \left[0_{ij}(mod\ n)\right] \quad (9)$$

also,

$$B*C = C*B$$
$$\Rightarrow \left[b_{ij}\right]*\left[c_{ij}\right] = \left[c_{ij}\right]*\left[b_{ij}\right]$$
$$\Rightarrow \left[\left(tb_{ij} + uc_{ij}\right)(mod\ n)\right] \cong \left[\left(tc_{ij} + ub_{ij}\right)(mod\ n)\right]$$

This implies that,

$$\left[\left(t\left(b_{ij} - c_{ij}\right) + u\left(c_{ij} - b_{ij}\right)\right)(mod\ n)\right] \cong \left[0_{ij}(mod\ n)\right] \quad (10)$$

as $n \mid [t(a_{ij}\text{-}b_{ij}) + u(b_{ij}\text{-}a_{ij})]$ and $n \mid [t(b_{ij}\text{-}c_{ij}) + u(c_{ij}\text{-}b_{ij})]$ by Equations (9) and (10)

$$n\left[\left[t\left(a_{ij} - b_{ij}\right) + u\left(b_{ij} - a_{ij}\right) + t\left(b_{ij} - c_{ij}\right) + u\left(c_{ij} - b_{ij}\right)\right]\right.$$
$$\Rightarrow n\left[\left[t\left(a_{ij} - c_{ij}\right) + u\left(c_{ij} - a_{ij}\right)\right]\right.$$
$$\Rightarrow \left[\left(t\left(a_{ij} - c_{ij}\right) + u\left(c_{ij} - a_{ij}\right)\right)(mod\ n)\right] \cong \left[0_{ij}(mod\ n)\right]$$
$$\Rightarrow \left[\left(\left(ta_{ij} + uc_{ij}\right)\left(tc_{ij} + ua_{ij}\right)\right)(mod\ n)\right] \cong \left[0_{ij}(mod\ n)\right]$$
$$\Rightarrow \left[\left(ta_{ij} + uc_{ij}\right)(mod\ n)\right] \cong \left[\left(tc_{ij} + ua_{ij}\right)(mod\ n)\right]$$
$$\Rightarrow \left[a_{ij}\right]*\left[c_{ij}\right] = \left[c_{ij}\right]*\left[a_{ij}\right] \Rightarrow A*C = C*A$$

Hence every $G_{nAG}(t,u)$ is a transitively commutative AG-groupoid.

Example-9. $G_{7AG}(5,4)$ is transitively commutative AG-groupoid.

Theorem-5. Every $G_{pAG}(t,u)$ is a cancellative AG-groupoid, if p is prime and $u \in Z_p - \{0\}$.

Proof. Let p is prime and $u \in Z_p - \{0\}$, to show that $G_{pAG}(t,u)$ is a cancellative AG-groupoid, we show that it is left cancellative AG-groupoid and right cancellative AG-groupoid.

For left cancellativity, let

$$A*X = A*Y \Rightarrow \left[a_{ij}\right]*\left[x_{ij}\right] = \left[a_{ij}\right]*\left[y_{ij}\right]$$
$$\Rightarrow \left[\left(ta_{ij} + ux_{ij}\right)(mod\ p)\right] \cong \left[\left(ta_{ij} + uy_{ij}\right)(mod\ p)\right]$$
$$\Rightarrow \left[u\left(x_{ij} - y_{ij}\right)(mod\ p)\right] \cong \left[0_{ij}(mod\ p)\right]$$
$$\Rightarrow p \left| u\left(x_{ij} - y_{ij}\right)\right.$$

Since as u is not devisible by p, because a non-zero u and $(x_{ij}-y_{ij})$ both are less than p, where p is prime. Therefore, $p \mid u(x_{ij}-y_{ij}) \Rightarrow [x_{ij} (\text{mod } p)] \cong [y_{ij} (\text{mod } p)] \Rightarrow X=Y$, and thus $G_{pAG}(t,u)$ is left cancellative. As every left cancellative AG-groupoid is the right cancellative AG-groupoid [17]. Hence, for prime p and $u \in Z_p - \{0\}$, $G_{pAG}(t,u)$ is a cancellative AG-groupoid.

Example-10. $G_{5AG}(3,4)$ is a cancellative AG-groupoid.

Theorem-6. An AG-groupoid of matrices over Z_n; $G_{nAG}(t,u)$ is an AG-band, if $t+u \cong 1 \pmod n$.

Proof. Let $t+u = 1$, to show that a matrix AG-groupoid $G_{nAG}(t,u)$ is an AG-band it is sufficient to show that $A*A=A$;

$$A*A = \left[a_{ij}\right]*\left[a_{ij}\right] \cong \left[\left(ta_{ij} + ua_{ij}\right)(mod\ n)\right]$$
$$\cong \left[(t+u)a_{ij}(mod\ n)\right]$$
$$\cong \left[a_{ij}(mod\ n)\right] as\ t+u \cong 1(mod\ n) \Rightarrow A*A = A$$

Hence a matrix AG-groupoid $G_{nAG}(t,u)$ is an AG-band, if $t+u \cong 1 \pmod n$.

Example-11. $G_{5AG}(2,4)$ is an AG-band.

4. EXISTENCE OF AG-GROUP OF MATRICES OVER Z_n

In this section, we introduce another class of groupoid of matrices as an AG-groups of matrices over Z_n. We study this AG-group of matrices over Z_n and obtain different results. The following theorem shows the existence of AG-groups of matrices over Z_n for $n \geq 3$, and indeed it gives a simple way of construction for matrix AG-groups (mod n) of any finite order.

The following theorem guarantees the existence of at least one AG-groups of matrices over Z_n for $n \geq 3$, if $t^2 \cong 1 (\text{mod } n)$.

Theorem-7. A groupoid of matrices over Z_n; $G_n(t,u)$ is an AG-groups of matrices over Z_n, if $t^2 \cong 1 (\text{mod } n)$ for $t \in Z_n - \{0\}$.

Proof. Given that a groupoid of matrices over Z_n; $G_n(t,u)$ satisfies $t^2 \cong 1 (\text{mod } n)$ for $t \in Z_n - \{0\}$, we have to show $G_n(t,u)$ is an AG-group of matrices over Z_n.

Left invertive law: We show that $(A*B)*C = (C*B)*A$, holds for all $A,B,C \in G_n(t,u)$, since

$$(A*B) * C \cong (t^2 a_{ij} + tb_{ij} + c_{ij}) (\text{mod } n) \quad (11)$$

and

$$(C*B) * A \cong (t^2 c_{ij} + tb_{ij} + a_{ij}) (\text{mod } n) \quad (12)$$

This implies that $G_n(t,u)$ is an AG-groupoid (mod n), as Equations (11 and 12) coincide for $t^2 \cong 1(\text{mod } n)$.

Nonassociativity: since

$$A^*(B^*C) \cong [(ta_{ij} + tb_{ij} + c_{ij}) (\text{mod } n)] \quad (13)$$

From Equations (11 and 13) it is clear that $G_n(t,u)$ is nonassociative in general.

Existence of left identity: '$0=[0_{ij}]$' is the left identity of $G_n(t,u)$;

$$0^*X \cong [x_{ij}(\text{mod } n)] = X; \forall X \in G_n(t,u)$$

but

$$X^*0 \cong [(tx_{ij})(\text{mod } n)] \neq X \text{ in general.}$$

Existence of inverses: $(n\text{-}1)tX$ or $-tX$ is the inverse of $X \forall X \in G_n(t,u)$;

$$(-tX)^*X = \left[-tx_{ij}\right]^*\left[x_{ij}\right] \cong \left[t\left(-tx_{ij} + x_{ij}\right)(mod\ n)\right]$$
$$\cong \left(-\left(t^2 - 1\right)x_{ij}\right)(mod\ n) \cong \left[0_{ij}(mod\ n)\right],$$

and

$$X^*(-tX) = \left[x_{ij}\right]^*\left[-tx_{ij}\right] \cong \left[\left(tx_{ij} + \left(-tx_{ij}\right)\right)(mod\ n)\right] \cong \left[0_{ij}(mod\ n)\right].$$

Hence $G_n(t,u)$ is an AG-group of matrices over Z_n.

For varying values of t we get different classes of AG-group of matrices over Z_n.

Corollary-3. Let $G_n(t,u)$ be a groupoid of matrices, then $G_n(1,1)$ is an AG-group of matrices over Z_n.

Proof. Since $(n\text{-}1)^2 \cong 1(\text{mod } n)$. The proof now follows by the Theorem 7.

5. CONCLUSION

In this paper a new class of AG-groupoids and AG-groups of modulo matrices over Z_n is investigated, moreover, various types of construction have been introduced for these AG-groupoids and AG-groups. Sufficient examples to show the existence of these notions are provided. It is to be noted that the provided examples are verified by various computer programs. The paper contains various nice results, the main result shows that a groupoid is an AG-groupoid of modulo matrices over Z_n, if $t^2 \cong u(\text{mod } n)$ for all $t,u \in Z_n$. Various other relations of these AG-groupoids and AG-groups of modulo matrices over Z_n with some of the already known classes of AG-groupoids are investigated. In Theorem-7, the class of AG-groupoids of modulo matrices over Z_n is further restricted to introduce AG-groups of modulo matrices over Z_n. In future, fuzzification of these AG-groupoids and AG-groups of modulo matrices over Z_n will be a nice work.

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