Mehran University Research Journal of Engineering & Technology
Vol. 37, No. 4, 481-490, October 2018
p-ISSN: 0254-7821, e-ISSN: 2413-7219
DOI: 10.22581/muet1982.1804.03

A Family of 6-Point *n*-Ary Interpolating Subdivision Schemes

ROBINA BASHIR*, AND GHULAM MUSTAFA*

RECEIVED ON 02.05.2017 ACCEPTED ON 29.05.2017

ABSTRACT

We derive three-step algorithm based on divided difference to generate a class of 6-point *n*-ary interpolating sub-division schemes. In this technique second order divided differences have been calculated at specific position and used to insert new vertices. Interpolating sub-division schemes are more attractive than approximating schemes in computer aided geometric designs because of their interpolation property. Polynomial generation and polynomial reproduction are attractive properties of sub-division schemes. Shape preserving properties are also significant tool in sub-division schemes. Further, some significant properties of ternary and quaternary sub-division schemes have been elaborated such as continuity, degree of polynomial generation, polynomial reproduction and approximation order. Furthermore, shape preserving property that is monotonicity is also derived. Moreover, the visual performance of proposed schemes has also been demonstrated through several examples.

Key Words: Interpolating Scheme, Continuity, Polynomial Generation and Reproduction, Monotonicity.

1. INTRODUCTION

Subdivision schemes are widely used in various branches of science such as CGGM (Computer Graphics, Geometric Modeling) and CAGD (Computer Aided Geometric Design). Subdivision scheme algorithms have ability to produce curves and surfaces usinga finit set of initial points. Due to interpolation properties, subdivision schemes became highly prominent compared to approximation schemes in geometric modeling.

Interpolating subdivision schemes were first developed by Dubuc [1]. Later on, Deslauriers and Dubuc [2] introduced a family of schemes by using the approach of cubic Lagrange interpolation polynomial. Dyn and co-workers recently proposed a family of 4-point interpolation subdivision schemes [3]. Lian [4] introduced 2m-point and 2m-point n-aryfor any $n \geq 2$ and odd $n \geq 3$, interpolation subdivision schemes for curve designing through wavelet theory. Mustafa and Khan [5] showed ternary 6-point interpolation subdivision scheme which used shape parameter for designing curves. A general formulae was introduced by Muatafa and Rehman [6] to cover (2b+4)-point *n*-ary interpolation and approximation schemes for every integers $b \geq 0$ and $n \geq 2$. Mustafa and Bashir [7] proposed a very potential but easy algorithm for producing 4-point *n*-ary interpolation scheme.

Authors E-Mail: (roba_bashir@hotmail.com, ghulam.mustafa@iub.edu.pk)
* Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur.

This is an open access article published by Mehran University Research Journal of Engineering and Technology, Jamshoro under the CC by 4.0 International License.

Mustafa et. al. [8] gave a clear method to constract odd points n-ary, for every odd $n \geq 3$, interpolation subdivision schemes. Both lower and higher arity schemes are generated by this formula. Bashir et. al. [9] introduced formulae of parametric and non-parametric bivariate subdivision scheme with four parameters.

In this paper, an efficient three-step technique is used to construct a group of 6-point n-ary interpolating subdivision schemes. Using this technique even-ary and odd-ary schemes can easily be generated. Monotonicity preservation has a significant role in shape preserving properties of subdivision schemes. Cai [10] offered a 4-point interpolation subdivision schemes, that is C^1 continuous and discussed the monotonicity preservation of the limit curve. Bari et. al. [11] presented $3n$-point quaternary shape preserving subdivision schemes.

Hussain and Hussain [12] presented a rational cubic function and used it to display monotone data into monotonic curves through producing certain restrictions for open factors in describing rational cubic function. Kuijt and Damme [13] introduced a group of interpolatory subdivision schemes, which preserve monotonic data. Shalom [14] presented a class of subdivision schemes with a finite support suitable for curve design and analyzed the monotonicity of the data. Recently, a novel binary 4-point subdivision schemes was given by Tan et. al. [15], which derived the monotonic preservation. Herein, in current study, we examined monotonicity preservation of ternary and quaternary subdivision schemes.

Present manuscript is prepared in the following heads: Section-2 introduced a 3-step algorithm which generate a family of n-ary interpolation subdivision schemes along with smoothness analysis of proposed schemes. Polynomial generation, polynomial reproduction and approximation order, monotonicity preservation of proposed ternary and quaternary subdivision schemes have been discussed in Section-3. Numerical examples have also been presented in the same part, and conclusion is presented in the last Section-4.

2. THREE-STEP ALGORITHM

In this section, we define the method for the construction of 6-point n-ary interpolating subdivision schemes by using three-step algorithm instead of using Lagrange polynomial and wavelet theory. These three steps are as follows:

Computation of the Second Divided Differences: At each old vertex, we calculate second divided difference R. R_a is the second divided difference at point a, i.e.

$$R_a = \frac{(b-a)-(a-z)}{n^2} = \frac{b-2a+z}{n^2} \tag{1}$$

Similarly, we can compute second divided differences, R_b, R_d and R_c at point b, c and d respectively, as shown in Fig. 1.

Computation of the New Second Divided Differences: By using the second divided differences, R_a, R_c, R_d the stencils of DD schemes, $A = [A_{L,1}, A_{L,2}, A_{L,3}, A_{L,4}]$, $B = [B_{S,1}, B_{S,2}, B_{S,3}, B_{S,4}]$ we calculate the new divided differences.

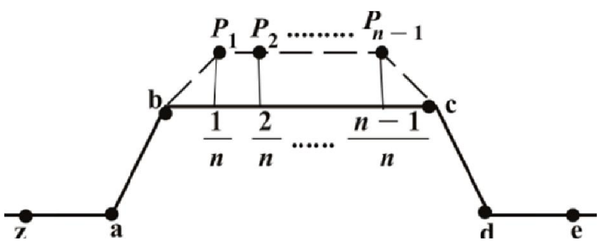


FIG. 1. LABELING OF A OPEN CONTROL POLYGON. $p_1, p_2, \ldots p_{n-1}$ ARE NEWLY INSERTED POINT BETWEEN OLD VERTICES b AND c

For odd a-ary scheme, we construct new divided differences as follows:

$$R_{pL}=A_{L,1}R_a+A_{L2}+A_{L,3}R_c+A_{L,4}R_d, R_{pv}=A_{n\text{-v},1}R_a+A_{n\text{-v}2}+A_{n\text{-v},3}R_c+A_{n\text{-v},4}R_d \tag{2}$$

For $n \geq 3$, $L = 1,2,\ldots,n - \mathrm{S}$, $V = S, \ldots., n - 1$ and $S = \dfrac{n+1}{2}$

In the case of even a-ary scheme, we use the above new divided differences, R_{pL} and R_{pV} have to construct another new divided difference, R_{pS} s.t

$$R_{pS} = B_{S,1}(R_{\mathrm{a}} + R_d) + B_{S,2}(R_b + R_c) \tag{3}$$

for $n \geq 2$, $L = 1,2,\ldots,S-1$, $V = S+1, \ldots., n - 1$ and $S = \dfrac{n+1}{2'}$

where

$$A_{L,1} = \frac{-L(n-L)(2n-L)}{6n^3}$$

$$A_{L,2} = \frac{(n^2-L^2)(2n-L)}{2n^3}$$

$$A_{L,3} = \frac{L(n-L)(2n-L)}{2n^3}$$

$$A_{L,4} = \frac{-L(n^2-L^2)}{6n^3}$$

and

$$B_{S,1} = \frac{-uS^2}{6n^3}$$

$$B_{S,2} = \frac{-u^2S}{2n^3}$$

Computation of Modified Vertices: By using Equations (2-3), we compute positions of modified vertices i.e. p_1, $p_2,\ldots,p_{n-1}$ by using the following:

$$R_{p1}=p_2-2_{p1}+b, R_{pi}=p_{i+1}-2_{pi}+p_{i\text{-}1}, R_{pn\text{-}1}=c-2p_{n\text{-}1}+\mathrm{p}_{n\text{-}2} \tag{4}$$

Where $i = 2,3,\ldots, n-2$.

2.1 Examples

Herein, it can be seen that 6-point n-ary interpolation subdivision schemes can be easily generated by above algorithm. In n-ary subdivision scheme each segment is divided into n sub-segments at each refinement level. One point is inserted at the position $1/n$,second point is inserted at the position $2/n$ and so on, (n-1) th point is inserted at the position n-1/n. By taking different values of n in Equations (1-4), we get different n-ary schemes.

For n=3 in Equations (1-2,4), the 6-point ternary interpolating scheme is:

$$\begin{aligned}
f_{3i}^{k+1} &= f_i^k, \\
f_{3i+1}^{k+1} &= \frac{14}{2187}f_{i-2}^k - \frac{178}{2187}f_{i-1}^k + \frac{1652}{2187}f_i^k + \frac{832}{2187}f_{i+1}^k - \frac{146}{2187}f_{i+2}^k + \frac{13}{2187}f_{i+3}^k, \\
f_{3i+2}^{k+1} &= \frac{13}{2187}f_{i-2}^k - \frac{146}{2187}f_{i-1}^k + \frac{832}{2187}f_i^k + \frac{1652}{2187}f_{i+1}^k - \frac{178}{2187}f_{i+2}^k + \frac{14}{2187}f_{i+3}^k,
\end{aligned} \tag{5}$$

For n=4 in Equations (1-4), we have following 6-point quaternary interpolating scheme

$$\begin{aligned}
f_{3i}^{k+1} &= f_i^k, \\
f_{4i+1}^{k+1} &= \frac{42}{8192}f_{i-2}^k - \frac{578}{2192}f_{i-1}^k + \frac{6820}{8192}f_i^k + \frac{300}{8192}f_{i+1}^k - \frac{430}{8192}f_{i+2}^k + \frac{38}{8192}f_{i+3}^k, \\
f_{4i+2}^{k+1} &= \frac{56}{8192}f_{i-2}^k - \frac{680}{8192}f_{i-1}^k + \frac{4720}{8192}f_i^k + \frac{4720}{8192}f_{i+1}^k - \frac{680}{8192}f_{i+2}^k + \frac{56}{8192}f_{i+3}^k, \\
f_{4i+2}^{k+1} &= \frac{38}{8192}f_{i-2}^k - \frac{430}{8192}f_{i-1}^k + \frac{2300}{8192}f_i^k + \frac{6820}{8192}f_{i+1}^k - \frac{578}{8192}f_{i+2}^k + \frac{42}{8192}f_{i+3}^k,
\end{aligned} \tag{6}$$

By substituting n=2 in Equations (1-4), we have the mask of 5-point binary interpolating scheme of [16].

2.2 Smoothness Analysis of Proposed Schemes

We discuss the analysis of a 6-point ternary and quaternary interpolating subdivision schemes. By using idea of [17] which helps to find convergence and smoothness of the schemes Equations (5-6), respectively.

Theorem 2.1: The 6-point ternary interpolating subdivision scheme Equation (5) is C^2

Proof. The Laurent polynomial $a(z)$ for the scheme Equation (5) is:

$$a(z)=\frac{1}{2182}\{13+14z^1-14z^3-146z^3-178z^4+832z^6+1652z^7+2187z^8+ \\ 1625z^8+1625z^9+832z^{10}-178z^{12}-1468z^{13}+14z^{15}+13z^{16}\} \tag{7}$$

Using Theorem 2.1 [18] for $n=3$ L=1 and $\beta=1,2$, we get

$$b^{[11]}(z)=\frac{1}{3}a_1(z)=\frac{1}{2187}\{13+z-14z^2-133z^3-31z^4+164z^5+699z^6+ \\ 789z^7+699z^9-31z^{10}-133z^{11}-14z^{12}+z^{13}+13z^{14}\} \tag{8}$$

and

$$b^{[11]}(z)=\frac{1}{3}a_2(z)=\frac{1}{729}\{13-12z-15z^2-106z^3+90z^4+180z^5+ \\ 429z^6+180z^7+90z^8-15z^{10}-12z^{11}+13z^{12}\} \tag{9}$$

If S_β is the scheme consequent to $a_\beta=(z)$ then by Theorem 2.1 [18]

$$\left\|\frac{1}{3}S_\beta\right\|_\infty=\max\left\{\sum_{j=Z}\left|b_{i+3,j}^{[\beta,1]}\right|:i=0,1,2\right\},\beta=1,2$$

Using Theorem 2.1 [18], Equations (8-9), we get

$$\left\|\frac{1}{3}S_1\right\|_\infty=\max\left\{\left|\frac{13}{2187}\right|+\left|\frac{-133}{2187}\right|+\left|\frac{699}{2187}\right|+\left|\frac{164}{2187}\right|+\left|\frac{-14}{2187}\right|,\left|\frac{1}{2187}\right|+\left|\frac{-31}{2187}\right|+\left|\frac{789}{2187}\right|+\left|\frac{-31}{2187}\right|+\left|\frac{1}{2187}\right|\right\}$$

and

$$\left\|\frac{1}{3}S_2\right\|_\infty=\max\left\{\left|\frac{13}{728}\right|+\left|\frac{-106}{729}\right|+\left|\frac{429}{729}\right|+\left|\frac{-106}{729}\right|+\left|\frac{13}{729}\right|,\left|\frac{-12}{729}\right|+\left|\frac{90}{729}\right|+\left|\frac{180}{729}\right|+\left|\frac{-15}{729}\right|\right\}$$

As we see

$$\left\|\frac{1}{3}S_1\right\|_\infty<1$$

then by Theorem 2.1 [18] the scheme Equation (5) is C^0.

Similarly $\left\|\frac{1}{3}S_2\right\|_\infty<1$

and

$$\left\|\left(\frac{1}{3}S_3\right)^6\right\|_\infty<1$$

then by Corollary 2.3 [19] the scheme Equation (5) is C^1 and respectively.

Theorem 2.2. The 6-point quaternary interpolating subdivision scheme Equation (6) is C^2

Proof of the above theorem is similar to the proof of Theorem 2.1.

3. PROPERTIES OF SUBDIVISION SCHEMES

Here, we discuss some important properties of subdivision schemes i.e. degree of polynomial generation, polynomial reproduction, approximation order of the schemes Equations (5-6).

Theorem 3.1. The degree of polynomial generation of the scheme Equation (5) is 3.

Proof. By using Equation (7), we have

$$a^{(0)}\left(\alpha_3^1\right)=a\left(e^{\frac{2\pi i}{3}}\right)=0 \text{ for } j=1 \text{ and } k=0$$

Similarly, we show that

$$a^{(k)}\left(\alpha_3^j\right)=0, \text{ for } j=1,2 \text{ and } k=0,1,2,3 \tag{10}$$

and

$$a^{(4)}\left(\alpha_3^j\right)\neq 0$$

where k denotes the order of derivative. Then by [20], degree of polynomial generation is 3.

Theorem 3.2. The degree of polynomial reproduction of the subdivision scheme Equation (5) is 3 with respect to the parameterizations of [20] if and only if

$$a^{(k)}(1) = 3\prod_{l=0}^{k-1}(\tau - l)$$

and

$$a^{(k)}\left(a_3^j\right) = 0 \text{ for } a_3^j = \exp\left(\frac{2\pi i}{3} j\right) \text{and } \tau = \frac{a'(1)}{3}$$

By taking first derivative of Equation (7) and put z+1 in it, we have

$$a^{(1)}(1) = 0$$

This implies that

$$\tau = \frac{a^{(1)}}{3} = 0$$

So from [20], the scheme Equation (5) has primal parameterization. From Equation (10), we have

$$a^{(k)}\left(a_3^j\right) = 0$$

By Equation (7), we get $a(1) = 3$.

Also

$$3\prod_{l=0}^{-1}(0 - l) = 3$$

which implies that

$$a(1) = 3\prod_{l=0}^{0-1}(\tau - l)$$

Similarly, for k=1,2,3, we can easily show that:

$$a^{(k)}(1) = 3\prod_{l=0}^{k-1}(\tau - l)$$

which completes the proof.

Since scheme Equation (5) reproduces polynomial of degree 3, so by using Theorem 2.5 [21], we get following theorem.

Theorem 3.3. A 6-point ternary interpolating subdivision scheme Equation (5) has an approximation order 4.

Theorem 3.4. The degree of polynomial generation of scheme Equation (6) is 3.

Proof of the above theorem is similar to the proof of Theorem 3.1.

Theorem 3.5. The degree of polynomial reproduction of the subdivision scheme Equation (6) is 3 with respect to the parameterizations of [20] if and only if:

$$a^{(k)}(1) = 4\prod_{l=0}^{k-1}(\tau - l)$$

and

$$a^{(k)}\left(\alpha_4^j\right) = 0, \text{ for } j = 1,2,3$$

for

$$k = 0,\ldots,3, \alpha_4^j = \exp\left(\frac{2\pi i}{4} j\right) \text{and } \tau = \frac{a'(1)}{4}$$

Proof of the above theorem is similar to the proof of Theorem 3.2.

Again by using Theorem 2.5 [21], we get following theorem.

Theorem 3.6. A 6-point quaternary interpolating subdivision scheme Equation (6) has an approximation order of 4.

3.1 Monotonicity Preservation

Definition 3.1. [22] A univariate data (x_i,f_i), i=0,1,2,…,n is monotonically increasing if $f_i<f_{i+1}\ \forall i=0,1,2,\ldots,n$ and the derivative at the data points obey the condition $d_i>0$ $\forall i=0,1,2,\ldots n$.

Here, we examine monotonicity preservation of 6-point ternary interpolating subdivision scheme Equation (5) and 6-point quaternary interpolating scheme Equation (6).

Theorem 3.7. Let be the sequence of initial points such that:

$$f_i^0 < f_{i+1}^0, i \in Z$$

Let

$$d_i^k = f_{i+1}^k - f_i^k, g_i^k = \frac{d_{i+1}^k}{d_i^k}, G^k = \max_i\left\{g_i^k, \frac{1}{g_i^k}\right\}, k \geq 0, k \in Z, i \in Z$$

Furthermore, let $0.2 \leq \lambda \leq 0.9$, $l \in$ R. If $\frac{1}{\lambda} \leq, G^0 \leq \lambda$, $\left\{f_i^k\right\}$ is defined by the subdivision scheme Equation (5), then

$$d_0^k > 0, \frac{1}{\lambda} \leq G^k \leq \lambda, k \geq 0, k \in Z, i \in Z \tag{11}$$

Proof. We use mathematical induction to prove Equation (11).

When $k=0, d_i^0 = f_{i+1}^k - f_i^0 > 0, \frac{1}{\lambda} \leq G^0 \leq \lambda$ then (11) is true.

Suppose that Equation (11) holds for k. i.e. $d_i^k = f_{i+1}^k - f_i^k > 0, \frac{1}{\lambda} \leq G^k \leq \lambda$, next we will prove that Equation (11) holds for $k+1$.

Since

$$d_{3i}^{k+1} = \frac{1}{2187}\left\{-14d_{i-1}^k + 164d_{i-1}^k + 699d_i^k - 133d_{i+1}^k + 13d_{i+2}^k\right\}$$

Similarly

$$d_{3i+1}^{k+1} = \frac{1}{2187}\left\{d_{i-2}^k - 31d_{i-1}^k + 789d_i^k - 31d_{i+1}^k + d_{i+2}^k\right\}$$

$$d_{3i+2}^{k+1} = \frac{1}{2187}\left\{13d_{i-2}^k - 133d_{i-1}^k + 699d_i^k + 164d_{i+1}^k - 14d_{i+2}^k\right\}$$

Next we show that

$$d_{3i}^{k+1} > 0, d_{3i+1}^{k+1} > 0, \text{and}\, d_{3i+2}^{k+1} > 0$$

Now

$$d_{3i}^{k+1} = \frac{1}{2187}\left\{-14\frac{1}{g_{i-2}^k}\frac{1}{g_{i-1}^k} + 164\frac{1}{g_{i-1}^k} + 699 - 133g_i^k + 13g_{i+1}^k g_i^k\right\}$$

This implies

$$d_{3i}^{k+1} \geq \frac{d_i^k}{2187}\left\{685 - 133\lambda + 164\frac{1}{\lambda} + 13\frac{1}{\lambda^2}\right\}$$

As we know that $d_i^k > 0$ and

$$\frac{1}{2187}\left\{685 - 133\lambda + 164\frac{1}{\lambda} + 13\frac{1}{\lambda^2}\right\} > 0$$

for $0.2 \leq \lambda \leq 0.9$.

This implies $d_{3i}^{k+1} > 0$.

Similarly, we see that $d_{3i+1}^{k+1} > 0$ and $d_{3i+2}^{k+1} > 0$ for $0.2 \leq \lambda \leq 0.9$.

Now we prove that:

$$\frac{1}{\lambda} \leq G^{k+1} \leq \lambda$$

first we show that

$$g_{3i}^{k+1} - \lambda \leq 0$$

Since

$$g_{3i}^{k+1} - \lambda = \frac{d_{3i+1}^{k+1}}{d_{3i}^{k+1}} - \lambda = \frac{\xi_1}{\xi_2}$$

where

$$\xi_1 = \{1 - 31g_{i-2}^k + 789g_{i-1}^k g_{i-2}^k - 31g_i^k g_{i-1}^k g_{i-2}^k + g_{i-1}^k g_i^k g_{i-1}^k g_{i-2}^k + 14\lambda - 164\lambda g_{i-2}^k$$
$$- 699g_{i-1}^k g_{i-2}^k + 133\lambda g_i^k g_{i-1}^k g_{i-2}^k - 13\lambda g_{i+1}^k g_i^k g_{i-1}^k g_{i-2}^k\}$$

and

$$\xi_2 = \{-14 + 164g_{i-2}^k + 699g_{i-1}^k g_{i-2}^k - 133g_i^k g_{i-1}^k g_{i-2}^k + 13\lambda g_{i-1}^k g_i^k g_{i-1}^k g_{i-2}^k\}$$

This implies that

$$g_{3i}^{k+1} - \lambda \le \frac{\{134\lambda^4 - 13\lambda^3 + 789\lambda^2 - 716\lambda - 163\}}{\{13\lambda^4 + 789\lambda^2 - 716\lambda - 163\}}$$

The denominator is positive and numerator is negative of above inequality for $0.2 \le \lambda \le 0.9$.

This implies that

$$g_{3i}^{k+1} - \lambda \le 0$$

This further implies $g_{3i}^{k+1} \le \lambda$. Now we show that

$$\frac{1}{g_{3i}^{k+1}} - \lambda \le 0.$$

For this consider

$$\frac{1}{g_{3i}^{k+1}} - \lambda = \frac{d_{3i}^{k+1}}{d_{3i+1}^{k+1}} - \lambda$$

This implies that:

$$\frac{1}{g_{3i}^{k+1}} - \lambda \le \lambda \frac{\frac{d_{i-1}^k}{2187}\{44\lambda^4 - \lambda^3 + 730\lambda^2 - 759\lambda - 14\}}{\frac{d_i^k}{2187}\left\{2\lambda^2 - 62\frac{1}{\lambda} + 789\right\}}$$

The denominator is positive and numerator is negative of above inequality for $0.2 \le \lambda \le 0.9$.

This implies that

$$\frac{1}{g_{3i}^{k+1}} - \lambda \le 0$$

This further implies $\frac{1}{g_{3i}^{k+1}} \le \lambda$. In the same way, we see that

$$g_{3i+1}^{k+1} \le \lambda, g_{3i+2}^{k+1} \le \lambda, \frac{1}{g_{3i+1}^{k+1}} \le \lambda \text{ and } \frac{1}{g_{3i+2}^{k+1}} \le \lambda$$

So $G^{k+1} \le \lambda$. Since $G^{k+1} = \max_i \left\{g_{i+1}^k, \frac{1}{g_{i+1}^k}\right\}$ it is obvious that $G^{k+1} \ge \frac{1}{\lambda}$.

Which completes the proof.

Similarly, we can prove the following theorem.

Theorem 3.8. Let $\{f_i^0\}_{i\in Z}$ be the sequence of initial points such that $f_i^0 <= f_{i+1}^k, i \in Z$. Let

$$d_i^k = f_{i+1}^k - f_i^k, g_i^k = \frac{d_{i+1}^k}{d_i^k}, G^k = \max_i \left\{g_i^k, \frac{1}{g_i^k}\right\}, k \ge 0, k \in Z, i \in Z$$

Furthermore, let $0.3 \le \lambda \le 0.9, \lambda \in R$. If $\frac{1}{\lambda} \le G^o \le \lambda, \{f_i^k\}$ is defined by the subdivision scheme Equation (6), then

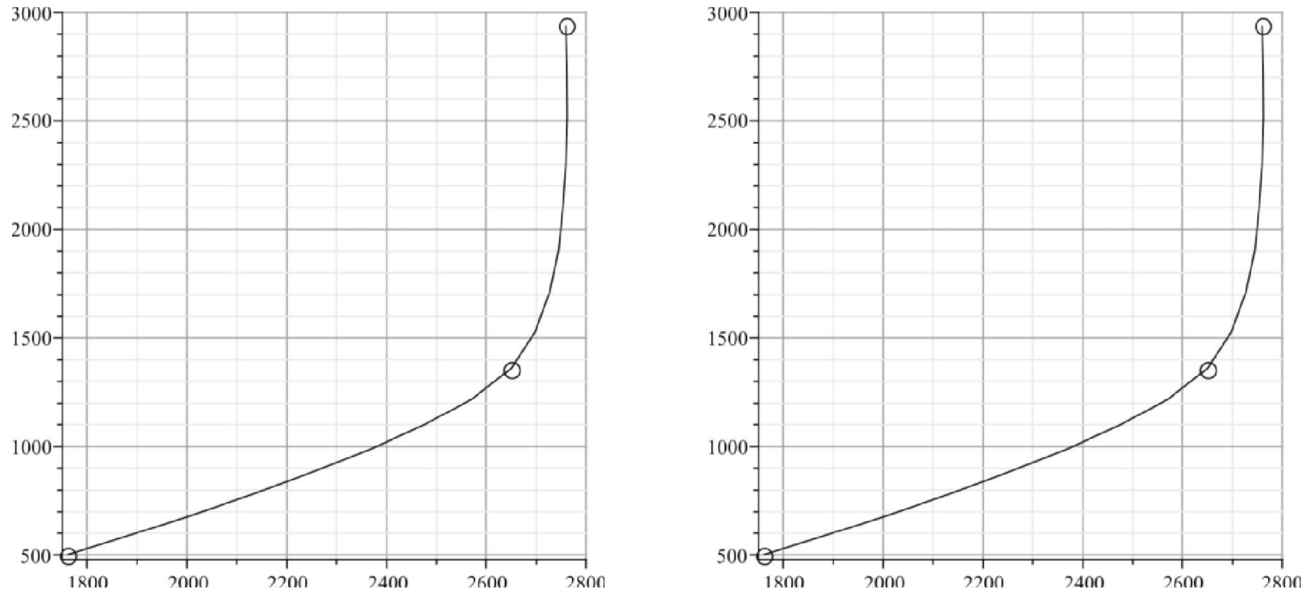
$$d_i^k > 0, \frac{1}{\lambda} \le G^k \le \lambda, k \ge 0, k \in Z, i \in Z \tag{12}$$

3.2 Numerical Examples

Fig. 2 is produced by using monotone data set given in Table 1. Monotone curves shown in Fig. 2(a-b) are produced by the schemes Equations (5-6) respectively. In Fig. 3, the initial control polygons are shown by dotted lines and solid lines show the limit curves. Limit curves presented in Fig. 3(a-b) and Table 2 are obtained by proposed schemes Equations (5-6) respectively.

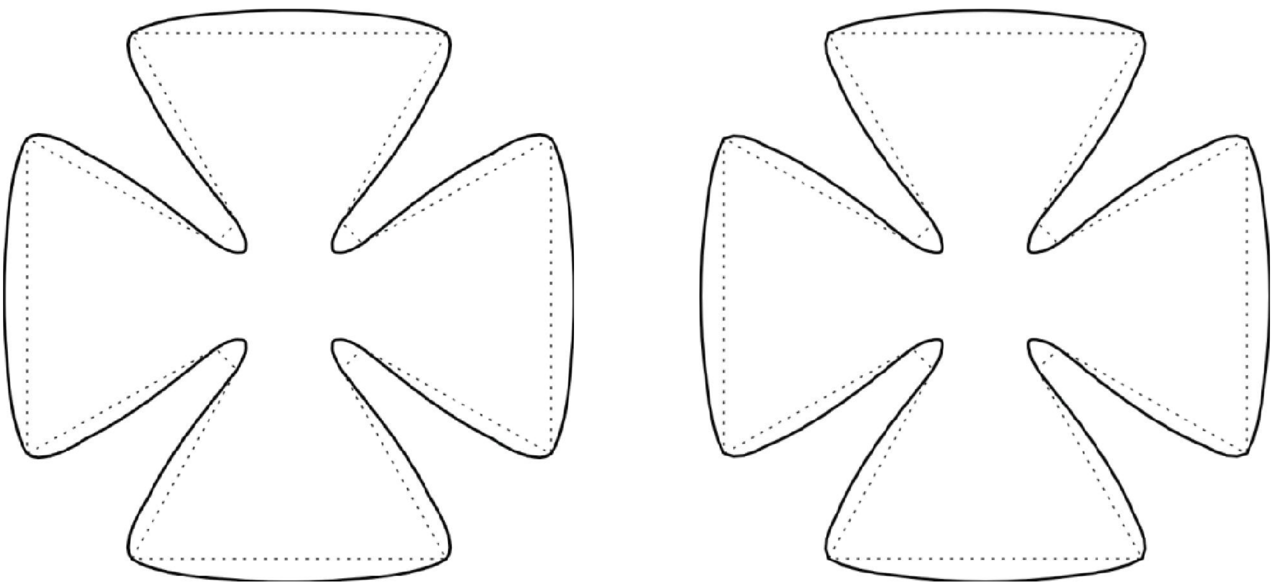
TABLE 1. MONOTONE DATA SET

X	1760	2650	2760
Y	500	1360	2940



(a) (b)

FIG. 2. THE CURVES (a) AND (b) ARE PRODUCED BY THE SCHEMES (5) AND (6), RESPECTIVELY, BY USING MONOTONE DATA SET



(a) (b)

FIG. 3(a-b). SHOW LIMIT CURVES OF THE SCHEMES (5) AND (6), RESPECTIVELY

TABLE 2. THE CONTINUITY, POLYNOMIAL GENERATION, POLYNOMIAL REPRODUCTION AND APPROXIMATION ORDER OF THE SCHEMES

Schemes	Continuity	Polynomial Generation	Polynomial Reproduction	Approximation Order
4-Point Interpolating [3]	1	3	3	4
5-Point Interpolating [23]	2	3	3	4
6-Point Interpolating [5]	2	3	3	4
Proposed 6-Point Interpolating	2	3	3	4

4. CONCLUSION

We have presented a simple and well-organized three-step algorithm which generates a group of 6-point n-ary interpolation subdivision schemes. Smoothness analysis of some proposed schemes has been carried out. Some important properties of proposed ternary and quaternary schemes like degree of polynomial generation, polynomial reproduction and approximation order have been discussed. Shape preserving property that is monotonicity preservation of data fitting has also been derived. Visual performance of proposed scheme is shown by several examples.

ACKNOWLEDGEMENT

Current study was accomplished by the Indigenous Ph.D. Scholarships Phase-II of Higher Education Commission, Pakistan and National Research Program for Universities, Grant No.-3183.

REFERENCES

[1] Dubuc, S., "Interpolation Through an Iterative Scheme", Journal of Mathematical Analysis & Applications, Volume 114, pp. 185-204, 1986.

[2] Deslauries, G., and Dubuc, S., "Symmetric Iterative Interpolation Processes", Constructive Approximation, Volume 5, pp. 49-68, 1989.

[3] Dyn, N., Levin, D., and Gregory, J.A., "A 4-Point Interpolatory Subdivision Scheme for Curve Design", Computer Aided Geometric Design, Volume 4, pp. 257-268, 1987.

[4] Lian, J.A., "On a-Ary Subdivision for Curve Design: III. 2m-Point and 2m+1-Point Interpolatory Schemes", Applications & Applied Mathematics: An International Journal, Volume 4, No. 1, pp. 434-444, 2009.

[5] Mustafa, G., and Khan, F., "Ternary 6-Point Interpolating Subdivision Scheme", Lobachevskii Journal of Mathematics, Volume 29, pp. 153-164, 2008.

[6] Mustafa, G., and Rehman, N.A., "The Mask of (2b+4)-Point n-Ary Subdivision Scheme", Computing, Volume 90, pp. 1-14, 2010.

[7] Mustafa, G., and Bashir, R., "Four-Point n-Ary Interpolating Subdivision Schemes", International Journal of Mathematics & Mathematical Sciences, pp. 8, 2013.

[8] Mustafa, G., Deng, J., Ashraf, P., and Rehman, N.A., "The Mask of Odd Points n-Ary Interpolating Subdivision Scheme", Journal of Applied Mathematics, pp. 20, 2012.

[9] Bashir, R., Bari, M., and Mustafa, G., "Generalization of Binary Tensor Product Schemes Depends Upon Four Parameters", Mehran University Research Journal of Engineering & Technology, Volume 37, No. 1, pp. 119-126, Jamshoro, Pakistan, January, 2018.

[10] Cai, Z., "Convergence, Error Estimation and Some Properties of Four-Point Interpolation Subdivision Scheme", Computer Aided Geometric Design, Volume 12, pp. 459-468, 1995.

[11] Bari, M., Bashir, R., and Mustafa, G., "3n-Point Quaternary Shape Preserving Subdivision Schemes", Mehran University Research Journal of Engineering & Technology, Volume 36, No. 3, Jamshoro, Pakistan, July, 2017.

[12] Hussain, M.Z., and Hussain, M., "Visualization of Data Preserving Monotonicity", Applied Mathematics & Computation, Volume 190, pp. 1353-1364, 2007.

[13] Kuijt, F., and Damme, R.V., "Monotonicity Preserving Interpolatory Subdivision Schemes", Journal of Computational & Applied Mathematics, Volume 101, pp. 203-229, 1999.

[14] Shalom, Y., "Monotonicity Preserving Subdivision Scheme", Journal of ApproximationTheory, Volume 74, pp. 41-58, 1993.

[15] Tan, J., Zhuang, X., and Zhang, L., "A New Four-Point Shape-Preserving C^3 Subdivision Scheme", Computer Aided Geometric Design, Volume 31, pp. 57-62, 2014.

[16] Augsdöefer, U.H., Dodgson, N.A., and Sabin, M.A., "Variation on the Four-Point Subdivision Scheme", Computer Aided Geometric Design, Volume 27, pp. 78-95, 2010.

[17] Dyn, N., and Levin, D., "Subdivision Scheme in the Geometric Modelling", Acta Numerica, Volume 11, pp. 73-144, 2002.

[18] Hassan, M.F., and Dodgson, N.A., "Ternary and Three-Point Univariate Subdivision Schemes", Curve and Surface Fitting, Cohen, A., Marrien, J.L., and Schumaker, L.L., (Editors), Sant-Malo, 2002, Nashboro Press, Brentwood, pp. 199-208, 2003.

[19] Aspert, N., "Non-Linear Subdivision of Univariate Signals and Discrete Surfaces", EPFL Thesis, Ecole Polytechnique Federale de Lausanne, Lausanne, Switzerland, 2003.

[20] Conti, C., and Hormann, K., "Polynomial Reproduction for Univariate Subdivision Scheme of any Arity", Approximation Theory, Volume 163, pp. 413-437, 2011.

[21] Dyn, N., "Tutorial on Multiresolution in Geometric Modelling Summer School Lecture Notes Series", Mathematics & Visualization, (Armin, I., Ewald, Q., and Michael, F.S., (Editors) Springer, [ISBN: 3-540-43639-1], 2002.

[22] Hussain, M.Z., Sarfraz, M., and Shaikh, T.S., "Monotone Data Visualization using RationalFunctions", World Applied Sciences Journal, Volume 16, No. 11, pp. 1496-1508, 2012.

[23] Mustafa, G., Irum, J., and Bari, M., "A New 5-Point Ternary Interpolating Subdivision Scheme and Its Differentiability", International Scholarly Research Network ISRN Computational Mathematics, pp. 10, 2012.