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On numerical schemes for determination of all roots simultaneously of non-linear equation

Nazir Ahmad Mir [a,b,*], Maria Anwar [a], Mudassir Shams [a], Naila Rafiq [b], Saima Akram [c]

[a] *Department of Mathematics and Statistics, Riphah International University, Islamabad Pakistan*

[b] *Department of Mathematics, National University of Modern Languages, Islamabad Pakistan*

[c] *Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan Pakistan*

[*] Corresponding author: Nazir Ahmad Mir, Email: nazirahmad.mir@gmail.com

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ABSTRACT

In this article, we first construct family of two-step optimal fourth order iterative methods for finding single root of non-linear equation. We then extend these methods for determining all the distinct as well as multiple roots of single variable non-linear equation simultaneously. Convergence analysis is presented for both the cases to show that the optimal order of convergence is 4 in case of single root finding method and 6 for simultaneous determination of all distinct as well as multiple roots of a non-linear equation. The computational cost, basins of attraction, computational efficiency, log of residual fall and numerical test functions validate that the newly constructed methods are more efficient as compared to the existing methods in the literature.

1. Introduction

Solving non-linear equation, $f(s) = 0,$ is the oldest problem of science in general and in particular in mathematics. These non-linear equations have diverse applications in many areas of science and engineering. In general, to find the roots of the equation, we look towards iterative schemes, which further are classified in to single and simultaneous root finding methods. This article addresses both types of iterative schemes. A variety of iterative methods having different convergence orders can be seen in the literature [7, 10, 16, 17, 22, 28, 31]. Iterative method which satisfies Kung and Traub conjecture is known as optimal as given in [26] and the efficiency index is defined by Ostrowski [25].

The afore-mentioned criteria are used for simple root finding algorithm, but mathematicians are also interested in finding of all roots of the non-liner equation simultaneously. This is due to the fact that simultaneous iterative methods are very popular due to their wider region of convergence and are more stable as compared to single root finding methods. The beauty of these methods lies in the fact that they are used in parallel computing which is more focused area presently. For more details, one can see [1-6, 9, 11-14, 19, 21, 29, 30] and references cited there in. The most famous of single root finding method is the classical Newton-Raphson method given by Eq. 1.

$$s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)} = (i = 1,2, \dots) \quad (1)$$

Method in Eq. 1 is optimal having convergence order 2 and efficiency index of 1.43 according to Kung and

Traub conjecture. We use Weierstrass' Correction [23] as follows.

$$\frac{f(s_i)}{f'(s_i)} = w(s_i) = \frac{f(s_i)}{\prod_{\substack{i\neq j \\ j=1}}^{n}(s_i-s_j)}, i, j = 1,2,3, \ldots, n \qquad (2)$$

In Eq. 1, to get classical Weierstrass-Dochive method which approximates all roots of the non-linear equation as follows.

$$s_{i+1} = s_i - \frac{f(s_i)}{\prod_{\substack{i\neq j \\ j=1}}^{n}(s_i-s_j)}. \qquad (3)$$

Method in Eq. 3 has convergence order 2. Later, Albert-Ehlirch presented 3rd. order simultaneous method given by Eq. 4.

$$s_{i+1} = s_i - \frac{1}{\frac{1}{N(s_i)} - \sum_{\substack{i\neq j \\ j=1}}^{n}\left(\frac{1}{s_i-s_j}\right)}, \qquad (4)$$

where $N(s_i) = \frac{f\prime(s_i)}{f(s_i)}$

The main aim of this paper is to construct family of optimal fourth order methods for determining single root of nonlinear equation without increasing the function values using weight function which will be a good addition of optimal methods in the literature. We, then further convert these methods into simultaneous iterative methods for finding all distinct as well as multiple roots of the non-linear equation. Using complex dynamical system, we are able to choose values of parameter used in the construction of iterative methods which give a wider convergence region.

2. Constructions of Single Root Finding Methods

King et al. [18] presented the following two-point optimal family of fourth order method (abbreviated as YM):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$

$$s_{i+1} = s_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{f(s_i)+\beta f(y_i)}{f(s_i)+(\beta-2)f(y_i)}\right), \qquad (5)$$

where $\beta\epsilon\Re$.

Chun et al. [20] presented the two-step fourth order optimal method (abbreviated as CM) as follows.

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$

$$s_{i+1} = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(1 + \left(\frac{f(y_i)}{f(s_i)}\right) + 2\left(\frac{f(y_i)}{f(s_i)}\right)^2\right). \qquad (6)$$

Jarrat et al. [26] gave the following fourth order optimal method as (abbreviated as JM) follows.

$$y_i = s_i - \frac{2}{3}\left(\frac{f(s_i)}{f'(s_i)}\right),$$

$$s_{i+1} = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(1 - \frac{2}{3}\left(\frac{f'(y_i)-f'(s_i)}{3f'(y_i)-f'(s_i)}\right)\right), \qquad (7)$$

Here, we propose the following family of iterative methods for solving single variable non-linear equation (1)

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$

$$z_i = y_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{1}{A-H(r)}\right), \qquad (8)$$

where $A - H(r)$ is the weight function using parameter A and $r = \left(\frac{f(y_i)}{f(s_i)}\right)$. It can be seen from the following theorem that the order of convergence of newly proposed Eq. 8 is four.

2.1 Theorem

Let ξϵI be single zero of a sufficiently differentiable function s: $I \rightarrow \Re$ for an open interval I. Let $e_i = s_i - \xi$ and $H(r)$ weight function with the conditions $H(0) = 1, H'(0) = 2$ and $A = 2, \beta = 1$. Then, order of convergence of the proposed method in Eq. 8 is 4 and error relation is given by Eq. 9.

$$e_{i+1} = (c_2^3 - c_2c_3)e_i^4 + O\left(e_i^5\right), \qquad (9)$$

where $c_k = \frac{s^k(\xi)}{k!s\prime(\xi)}, k = 2,3, \ldots$

2.2 Proof

Let $e_i = s_i - \xi$ (10)

Using Taylor series, we have Eqs. 11 and 12.

$$f(s_i) = f'(\xi)\left(e_i + c_2e_i^2 + c_3e_i^3 + c_4e_i^4\right) + O\left(e_i^5\right) \quad (11)$$

$$f'(s_i) = f'(\xi)\left(1 + 2c_2e_i + 3c_3e_i^2 + 4c_4e_i^3\right) + O\left(e_i^4\right). \quad \ldots(12)$$

Dividing Eq. 11 by Eq. 12, we get Eq. 13.

$$\frac{f(s_i)}{f\prime(s_i)} = e_i + c_2e_i^2 + (-2c_3 + 2c_2^2)e_i^3 + \cdots \qquad (13)$$

Now,

$$e_i = c_2e_i^2 - (-2c_3 + 2c_2^2)e_i^3 - (3c_4 + 7c_2c_3 - 4c_2^3)e_i^4 + O\left(e_i^5\right) \qquad (14)$$

$$f(y_i) = f'(\xi)\left(c_2e_i^2 + (-2c_3 + 2c_2^2)e_i^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_i^4\right) + \cdots \qquad (15)$$

Dividing Eq. 15 by Eq. 12, we have Eq. 16.

$$\frac{f(y_i)}{f'(s_i)} = c_2e_i^2 + (2c_3 - 4c_2^2)e_i^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_i^4 + \cdots \qquad (16)$$

Expanding $H(r)$ about origin, we have Eq. 17.

$$H(r) = H(0) + H'(0)r + H''(0)\frac{r^2}{2!} + \cdots \tag{17}$$

where $r = \frac{f(y_i)}{f(s_i)}$.

Now,

$$\frac{1}{A-H(r)} = \frac{1}{A-H(0)} + \frac{H(0)c_2}{(A-H(0))^2}e_i + \frac{\frac{-3H'(0)c_2^2-2H'(0)c_3}{A-H(0)} + \frac{(H'(0))^2c_2^2}{(A-H(0))^2}}{A-H(0)}e_i^2 + \cdots \tag{18}$$

where $z_i = y_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{1}{A-H(r)}\right)$,

$$= \left(c_2 - \frac{\beta c_2}{A-H(r)}\right)e_i^2 + \left(-2c_2^2 + 2c_3 - \left(\frac{\beta H'(0)c_2^2}{(A-H(r))^2} + \frac{4\beta c_2^2}{A-H(r)} - \frac{2\beta c_3}{A-H(r)}\right)\right)e_i^3 + \cdots \tag{19}$$

Using the value $H(0) = 1, \beta = 1, A = 2$ in Eq. 19, we have Eq. 20.

$$e_{i+1} = (2c_2^2 - H'(0)c_2^2)e_i^3 + \cdots \tag{20}$$

Now, using $H'^{(0)} = 2$ in Eq. 20, we get Eq. 21.

$$e_{i+1} = (c_2^3 - c_2c_3)e_i^4 + O(e_i^5) \tag{21}$$

Hence, it proves the theorem.

3. Concrete Methods

Here, we discuss some concrete optimal fourth order methods.

3.1 Construction of Method MS1

Let, $H(r) = 1 + 2r, where\ H(0) = 1, H'(0) = 2.$ Thus, we have the following optimal fourth order method.

$$z_i = y_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{1}{2-(1+2r)}\right), \tag{22}$$

where $y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)$ and $r = \left(\frac{f(y_i)}{f(s_i)}\right)$.

3.2 Construction of Method MS2

Let $H(r) = \frac{2+r}{2-r} + r\ where\ H(0) = 1, H'(0) = 2.$ Thus, we have the following optimal fourth order method.

$$z_i = y_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{1}{2-\left(\frac{2+r}{2-r}+r\right)}\right), \tag{23}$$

where $y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)$ and $r = \left(\frac{f(y_i)}{f(s_i)}\right)$

3.3 Construction of Method MS3

Let $H(r) = 1 + \frac{2r}{1+r^2},\ where\ H(0) = 1, H'(0) = 2.$ Thus, we have the following optimal fourth order method.

$$z_i = y_i - \left(\frac{f(y_i)}{f'(s_i)}\right)\left(\frac{1}{2-\left(1+\frac{2r}{1+r^2}\right)}\right), \tag{24}$$

where $y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)$ and $r = \left(\frac{f(y_i)}{f(s_i)}\right)$.

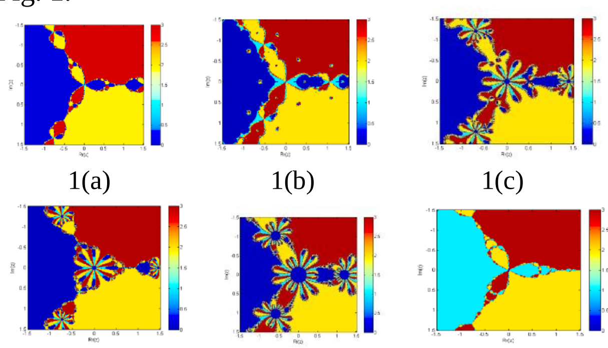
Thus, we have constructed here new methods Eq. 22, Eq. 23 and Eq. 24 abbreviated as (MS1, MS2 and MS3).

4. Complex Dynamical Study of Families of Iterative Methods

Here, we discuss stability of family of iterative method (MS1-MS3, KM, CM, and JM) only in the background contexture of complex dynamics. Recalling some basic concepts of this theory (detail information can be found in [10, 17, 24, 27]). Taking a rational function $R: C \to C$, where C denotes the Riemann sphere, the orbit $s_0 \in C$ defines a set such as $orb(s) = \{s_0, R_f(s_0), \ldots, R_f^m(s_0)\}$. An attracting point defines basin of attraction, $R_f^m(s^*)$ as the set of starting points whose orbit tends to s^*.

Further, the implementation of the dynamical plane of rational operator corresponding to iterative methods divides the complex plane into a mesh of values of real part along x-axis and imaginary part along y-axis. The initial estimates is depicted in a colour depending on where its orbit converges and thus basins of attraction of corresponding iterative methods are obtained.

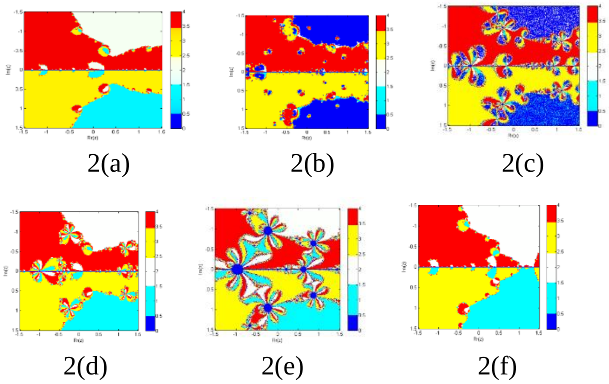
Here, we find the basins of attraction for non-linear equations. We use the mesh of [400×400]. Divergence is represented by black colour. We use absolute error value as a stopping criteria and maximum number of iterations are taken as 25. If the method converges to a root then a specific colour is assigned to it. Let us consider the basins of attraction for non-linear equation, Fig. 1.



1(a) 1(b) 1(c)

1(d) 1(e) 1(f)

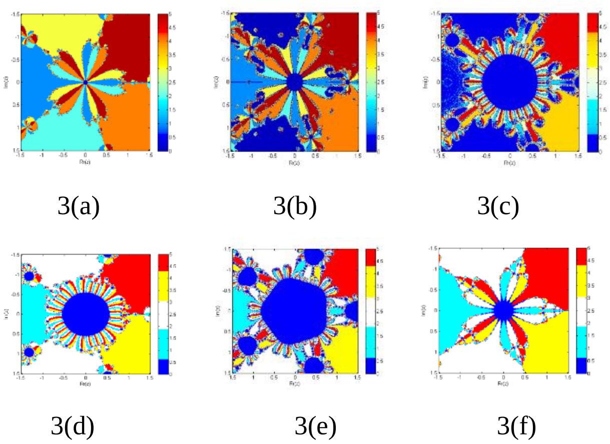
Fig. 1. (a-f), shows the basins of attraction of methods MS1-MS3, KM, CM and JM respectively for non-linear function f_1 (s)=s^3+1having roots1,0.5+0.8i,0.5-0.8i. Brightness in colour shows less number of iteration for approximating the roots



2(a) 2(b) 2(c)

2(d) 2(e) 2(f)

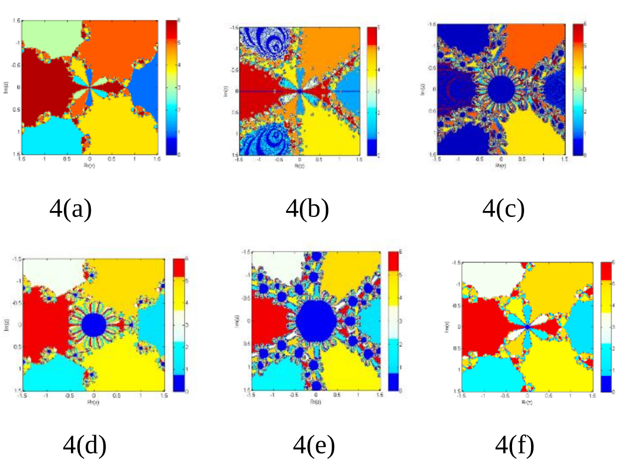
Fig. 2. (a-f), shows the basins of attraction of method MS1-MS3, KM, CM and JM) respectively for non-linear function f_2 (s)=s^4+2s^3+3s^2+4s+5 having roots 0.2+1.4i,0.2-1.4i,-1.2+0.8i,-1.2-0.8i. Brightness in colour shows less number of iteration for approximating the roots



3(a) 3(b) 3(c)

3(d) 3(e) 3(f)

Fig. 3. (a-f), shows the basins of attraction of method (MS1-MS3, KM, CM, JM) respectively for non-linear function f_3 (s)=s^5+5 having roots-1.3,-0.4+1.3i,-0.4-1.3i,1.1+0.8i,1.1-0.8i. Brightness in colour shows less number of iteration for approximating the roots



4(a) 4(b) 4(c)

4(d) 4(e) 4(f)

Fig. 4. (a-f), shows the basins of attraction of methods MS1-MS3, KM, CM and JM) respectively for non-linear function f_4 (s)=s^6-s^3-1 having roots 1.1,0.8,-0.5+1.01i,-0.5-1.01i,0.4+0.7i,0.4-0.7i. Brightness in colour shows less number of iteration for approximating the roots

5. Generalization to Simultaneous Iterative Methods

Consider non-linear equation having n roots, then $f(s)\ and\ f'(s)$ can be approximated as Eqs. 25 and 26.

$$f(s) = \prod_{j=1}^{n}(s - s_i)$$

$$f'(s) = \sum_{j=1}^{n}\prod_{j=1}^{n}(s - s_i) \tag{25}$$

$$\frac{f\prime(s_i)}{f(s_i)} = \frac{1}{(s-s_j)} + \sum_{\substack{i\neq j\\ j=1}}^{n}\frac{1}{s_i - s_j} = \frac{1}{\frac{1}{s-s_i} - \sum_{\substack{i\neq j\\ j=1}}^{n}\frac{1}{s_i-s_j}} \tag{26}$$

This gives Eqs. 27 and 28.

$$s_{i+1} = s_i - \frac{1}{\frac{1}{N(s_i)} - \sum_{\substack{i\neq j\\ j=1}}^{n}\left(\frac{1}{s_i - s_j}\right)} \tag{27}$$

$$N(s_i) = \frac{f\prime(s_i)}{f(s_i)}. \tag{28}$$

Now, from (27), an approximation of $\frac{f\prime(s_i)}{f(s_i)}$ is formed by replacing $s_j\ with\ z_{tj}$, we get Eq. 29.

$$\frac{f\prime(s_i)}{f(s_i)} = \frac{1}{\frac{1}{N(s_i)} - \sum_{\substack{i\neq j\\ j=1}}^{n}\left(\frac{1}{s_i - z_{tj}}\right)}, t = 1,2,3, \tag{29}$$

where $z_{tj}, t = 1,2,3$ are given in Eqs. 22-24 respectively.

Using Eq. 29 in Eq. 1, we get Eq. 30.

$$s_{i+1} = s_i - \frac{1}{\frac{1}{N(s_i)} - \sum_{\substack{i\neq j\\ j=1}}^{n}\left(\frac{1}{s_i - z_{tj}}\right)}, (t = 1,2,3) \tag{30}$$

In case of multiple roots, we have Eq. 31.

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{1}{N(s_i)} - \sum_{\substack{i\neq j\\ j=1}}^{n}\left(\frac{\sigma_j}{s_i - z_{tj}}\right)}, (t = 1,2,3) \tag{31}$$

where

$$z_{1j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\frac{1}{2-(1+2r1)}\right),$$

$$z_{3j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\frac{1}{2-\left(\frac{2+r1}{2-r1}+r1\right)}\right),$$

$$z_{3j} = y_j - \left(\frac{f(y_j)}{f'(s_j)}\right)\left(\frac{1}{2-\left(\frac{2+r1}{2-r1}+r1\right)}\right),$$

and $y_j = s_j - \left(\frac{f(s_j)}{f'(s_j)}\right), r1 = \left(\frac{f(y_j)}{f(s_j)}\right).$

6. Convergence of Simultaneous Method of Order Six

Using correction z_{tj}, $t = 1,2,3$, we get the following three simultaneous iterative methods for extracting distinct as well as multiple roots of non-linear equation as Eq. 32.

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{1}{N(s_i)} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{\sigma_j}{s_i - z_{tj}}\right)}, (t = 1,2,3) \tag{32}$$

where $i, j = 1, ,3, \ldots, n$.

Thus, we have presented three simultaneous iterative methods for $t = 1,2,3$ abbreviated as M1, M2, M3 respectively.

7. Convergence Analysis

In this section, the convergence analysis of a family of simultaneous methods $(M1-M3)$, for multiple roots is given in the form of the following theorem. Obviously, convergence for distinct roots will follow from the convergence of the theorem when the multiplicities of the roots are simple.

7.1 Theorem

Let $\xi_1, \xi_2, \xi_{3,\ldots,}\xi_n$ be the n number of simple roots of the non-linear equation. *If* $s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, \ldots, s_n^{(0)}$, be the initial approximations of the roots respectively and sufficiently close to actual roots, the order of convergence of methods (M1-M3) equals six.

7.2 Proof

Let $\varepsilon_i = s_i - \xi_i \; and \; \varepsilon'_i = s_{i+1} - \xi_i$ be the errors in s_i and s_{i+1} approximations respectively.

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{1}{N(s_i)} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{\sigma_j}{s_i - z_{tj}}\right)} \tag{33}$$

$$f(s) = \prod_{j=1}^{n}(s - s_i), \tag{34}$$

Then obviously for distinct roots, we have Eq. 35.

$$\frac{1}{N(s_i)} = \frac{f(s_i)}{f'(s_i)} = \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{1}{s_i - \xi_j}\right) = \frac{1}{s_i - \xi_i} + \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{1}{s_i - \xi_j}\right) \tag{35}$$

Then for the multiple roots we have Eqs. 36-40.

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{\sigma_i}{s_i - \xi_j} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{\sigma_i}{s_i - \xi_i}\right) - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{\sigma_j}{s_i - z_{tj}}\right)} \tag{36}$$

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{\sigma_i}{s_i - \xi_j} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{\sigma_i(s_i - z_{tj} - s_i - \xi_j)}{(s_i - \xi_j)(s_i - z_{tj})}\right)} \tag{37}$$

$$s_{i+1} = s_i - \frac{\sigma_i}{\frac{\sigma_i}{s_i - \xi_j} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{-\sigma_i(z_{tj} - \xi_j)}{(s_i - \xi_j)(s_i - z_{tj})}\right)} \tag{38}$$

$$s_{i+1} - \xi_i = s_i - \xi_i - \frac{\sigma_i}{\frac{\sigma_i}{s_i - \xi_j} - \sum_{\substack{i \neq j \\ j=1}}^{n} \left(\frac{-\sigma_i(z_{tj} - \xi_j)}{(s_i - \xi_j)(s_i - z_{tj})}\right)} \tag{39}$$

$$\varepsilon'_i = \varepsilon_i - \frac{\sigma_i}{\frac{1}{\varepsilon_i} - \sum_{\substack{i \neq j \\ j=1}}^{n} \varepsilon_i L_{ij}}, \tag{40}$$

where $L_{ij} = \frac{-\sigma_j}{(s_i - \xi_j)(s_i - z_{tj})}$ and $z_{tj} - \xi_j = \varepsilon_j^4$ from Eq. 21. Thus,

$$\varepsilon'_i = \varepsilon_i - \frac{\sigma_i \varepsilon_i}{1 - \varepsilon_i \sum_{\substack{i \neq j \\ j=1}}^{n} \varepsilon_i L_{ij}} \tag{41}$$

$$\varepsilon'_i = \frac{\varepsilon_i - \varepsilon_i^2 \sum_{\substack{i \neq j \\ j=1}}^{n} \varepsilon_i^2 L_{ij - \varepsilon_i}}{1 - \varepsilon_i \sum_{\substack{i \neq j \\ j=1}}^{n} \varepsilon_i L_{ij}} \tag{42}$$

Since the errors ε_{js} , are very small and due to convergence order of the method, we can assume that absolute values of all errors $\varepsilon_j, (j = 1,2,3, \ldots)$ are of the same order, say $|\varepsilon_j| = O|\varepsilon_i|$. For further detail, see the reference ([9]). Therefore, from Eq. 41, we have Eqs. 43-44.

$$\varepsilon'_i = \varepsilon_i^2 (O(\varepsilon_i^4)) \tag{43}$$

$$\varepsilon'_i = O(\varepsilon_i^6) \tag{44}$$

Hence, the theorem is proved.

8. Computational Aspect

Here, we compare the computational efficiency of the M. S. Petković, L. Rančić, M. R. Milošević method [8] and the new methods (M1-M3). As presented in [8], the efficiency of an iterative method can be estimated using the efficiency index given by

$$EL(M) = \frac{Log u}{D} \tag{45}$$

where *D* is the computational cost and *u* is the order of convergence of the iterative method. Using arithmetic operation per iteration with certain weight depending on the execution time of operation, we evaluate the computational cost *D*. The weights used for division, multiplication and addition plus subtraction are w_d, w_m, w_{as} respectively. For a given polynomial of degree m and n roots, the number of division, multiplication addition and subtraction per iteration for all roots are denoted by $D_m, M_m \; and \; AS_m$. The cost of computation can be calculated as:

$$D = D(M) = w_{as} AS_m + w_m M_m + w_d D_m. \tag{46}$$

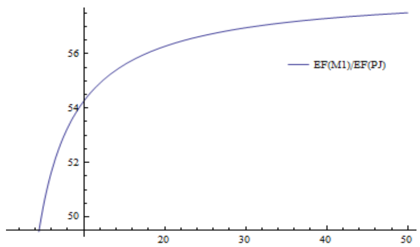
Thus, Eq. 46 becomes Eq. 47.

$$EL(M) = \frac{Log u}{w_{as} AS_m + w_m M_m + w_d D_m} \tag{47}$$

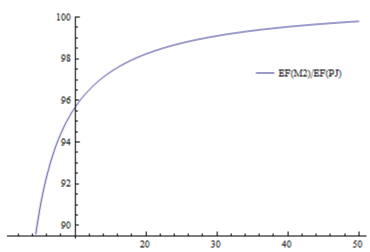
Considering the number of operations of a complex polynomial with real and complex roots reduce to operation of real arithmetic, given in Table 1 as polynomial degree m taking the dominant term of order (m^2). Applying Eq. 45 and data given in Table 1, we calculate the percentage ratio $\rho(M1-M3),X)$ [8] given by Eq. 48.

$$\rho(M1-M3),X)=\left(\frac{EL(M1-M3)}{EL(X)}-1\right)\times 100 \qquad (48)$$

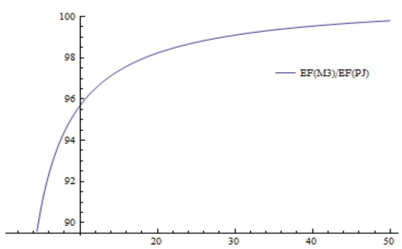
where X is Petkovic method [8] of order 4. Figure 5(a-d) graphically illustrates these percentage ratios. It is evident from Figure 5(a-d) that the newly constructed simultaneous methods (M1-M3) are more efficient as compared to Petkovic method [8].



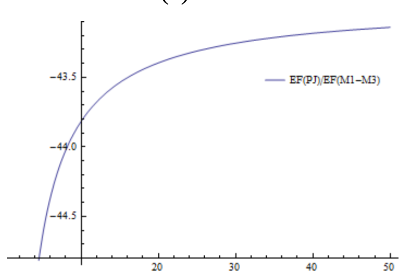
5(a)



5(b)



5(c)



5(d)

Fig. 5. (a-d), shows computational efficiency of methods (M1-M3) w.r.t M.S. Petković, L. Rančić, M. R. Milošević method respectively.

We also calculate the CPU execution time, as all the calculations are done using maple 18 on a Processor Intel(R) Core(TM) i3-3110m CPU@2.4GHz with 64-bit Operating System. We observe from Tables that CPU time of the methods M1-M3 is less than M.S. Petković, et al method [8], showing the dominance efficiency of our methods (M1-M3) as compared to them.

Table 1

Number of basic arithmetic operations

Methods	CO	AS_m	M_m	D_m
M1	6	$7m^2+O(m)$	$1m^2+O(m)$	$2m^2+O(m)$
M2	6	$8m^2+O(m)$	$1m^2+O(m)$	$2m^2+O(m)$
M3	6	$8m^2+O(m)$	$2m^2+O(m)$	$2m^2+O(m)$
PJ	6	$8m^2+O(m)$	$6m^2+O(m)$	$2m^2+O(m)$

9. Numerical Results

Here, some numerical examples are considered in order to demonstrate the performance of our family of two-step optimal fourth order single root finding methods (MS1-MS3) and sixth order simultaneous methods (M1-M3) respectively. We compare our family of single root finding optimal fourth order methods (M1-M3) with optimal fourth order methods (KM, CM) methods. Family of simultaneous methods of order six is compared with M.S. Petković, et al method [8] of same order (abbreviated as PJ method). All the computations are performed using MAPLE 18 with 2500 (64 digits floating point arithmetic in case of simultaneous methods) significant digits with stopping criteria as follows.

1. $e_i=|f(s_i)|<\epsilon,$
2. $e_i=|s_i-\alpha|<\epsilon$

where e_i represents the absolute error of function values in norm-2. We take ϵ^{-600} for single root finding method and ϵ^{-30} for simultaneous determination of all roots of non-linear equation (1).

Numerical tests examples from [14-15, 30-31] are provided in Tables 2-8 and 9. In Tables 2, 4, 6, 8 the stopping criterion 2 is used while in Tables 3, 5, 7 and 9 stopping criteria 1 and 2 both are used. In all tables, CO represents the convergence order, n represents the number of iterations, ρ represents computational order of convergence [15], $\gamma=1$ represents all distinct roots, $\gamma\neq 1$ represents multiple roots and CPU represents computational time in seconds. We observe that numerical results of the methods (in case of single root finding methods MS1-MS3) as well as simultaneous determination (M1-M3 of all roots) are better than KM, CM, JM and PJ respectively on same number of iterations. Figure 6(a-k) represents the residual falls for the iterative methods (MS1-MS3, KM, CM, JM and PJ).

9.1 Example 1: Beam Designing Model (Application in Engineering) [31]

An engineer considers a problem of embedment t of a sheet-pile wall resulting in a non-linear function, given as Eq. 49.

$$f_5(s) = \frac{s^3+2.87s^2-10.28}{4.62} - s \tag{49}$$

The exact roots of Eq. 49 are as follows.

$$\xi_1 = 2.0021, \xi_2 = -3.3304, \xi_3 = -1.5417$$

The initial estimates are taken as follows.

$$s_1^{(0)} = 2.5, s_2^{(0)} = -7.4641, s_3^{(0)} = -0.5359$$

Table 2

Simultaneous determination of all roots

$f_5(s) = \frac{s^3+2.87s^2-10.28}{4.62} - s$

$\xi_1 = 2.0021, \xi_2 = -3.3304, \xi_3 = -1.5417$

$s_1^{(0)} = 2.5, s_2^{(0)} = -7.4641, s_3^{(0)} = -0.5359$

Method	CO	CPU	γ	n	e_1	e_2	e_3
PJ	6	0.063	$\gamma = 1$	4	5.1e-21	5.5e-20	2.2e-20
M1-M4	6	0.032	$\gamma = 1$	4	6.7e-20	1.0e-20	4.1e-21

Table 3

Comparison of single roots finding methods

$f_5(s) = \frac{s^3+2.87s^2-10.28}{4.62} - s$, $s_0 = 2.5$, $\alpha = 2$,CO=4

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
MS1	6.9e-99	3.5e-87	0.032	4.0
MS2	5.7e-74	2.2e-93	0.063	4.0
MS3	1.7e-68	9.0e-87	0.047	4.01
KM	1.6e-67	8.5e-86	0.031	4.0
CM	3.2e-65	1.6e-83	0.047	4.0
JM	1.0e-40	1.6e-49	0.016	3.9

9.2 Example 2 [15]

Here, we consider another standard test function for the demonstration of convergence behaviour of newly constructed methods.

Consider,

$$f_6(s) = sin^3\left(\frac{s-1}{2}\right) sin^3\left(\frac{s-2}{2}\right) sin^3\left(\frac{s-2.5}{2}\right) \tag{50}$$

with multiple exact roots ($\gamma \neq 1$).

$$\xi_1 = 1, \xi_2 = 2, \xi_3 = 2.5$$

The initial guessed values have been taken as follows.

$$s_1^{(0)} = -0.2, s_2^{(0)} = 1.7, s_3^{(0)} = 3$$

For distinct roots (γ =1), we have the following.

$$f_6(s) = sin\left(\frac{s-1}{2}\right) sin\left(\frac{s-2}{2}\right) sin\left(\frac{s-2.5}{2}\right)$$

Table 4

Simultaneous determination of all roots

$f_6(s) = sin^3\left(\frac{s-1}{2}\right) sin^3\left(\frac{s-2}{2}\right) sin^3\left(\frac{s-2.5}{2}\right)$,

$\xi_1 = 1, \xi_2 = 2, \xi_3 = 2.5$

$s_1^{(0)} = -0.2, s_2^{(0)} = 1.7, s_3^{(0)} = 3$

Method	CO	CPU	γ	n	e_1	e_2	e_3
PJ	6	0.063	$\gamma = 1$	5	0.019	4.6e-3	9.9e-10
PJ	6	0.109	$\gamma \neq 1$	5	2.9e-9	1.7e-4	1.9e-9
M1-M3	6	0.063	$\gamma = 1$	5	2.6e-13	2.1e-7	9.5e-10
M1-M3	6	0.109	$\gamma \neq 1$	5	6.1e-2	6.8e-8	7.5e-9

Table 5

Comparison of single roots finding methods

$f_6(s) = sin\left(\frac{s-1}{2}\right) sin\left(\frac{s-2}{2}\right) sin\left(\frac{s-2.5}{2}\right)$, $s_0 = 0.5$, $\alpha = 1$,CO=4

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
MS1	2.5e-550	8.3e-2200	0.860	4.0
MS2	1.1e-558	2.5e-2233	0.875	4.0
MS3	8.8e-551	1.1e-2201	0.890	4.0
KM	1.2e-536	7.5e-2145	0.859	4.0
CM	8.0e-507	1.9e-2025	0.906	4.0
JM	3.7e-550	3.6e-2199	1.282	4.0

9.3 Example 3 [14]

Here, we consider another standard test function for the demonstration of convergence behaviour of newly constructed methods.

Consider,

$$f_7(s) = e^{s(s-1)^2(s-2)^2(s-3)^3}, \tag{51}$$

with multiple exact roots ($\gamma \neq 1$), $\xi_1 = 0, \xi_2 = 1, \xi_3 = 2, \xi_4 = 2$. The initial guessed values have been taken as follows.

$$s_1^{(0)} = 0.1, s_2^{(0)} = 0.9, s_3^{(0)} = 1.8, s_4^{(0)} = 2.9$$

For distinct roots (γ =1), $f_7(s) = e^{s(s-1)(s-2)(s-3)}$.

9.4 Example 4 [14]

Here, we consider another standard test function for the demonstration of convergence behaviour of newly constructed methods.

Consider

$$f_8(s) = sinh^5\left(\frac{s+2}{2}\right) sinh^6\left(\frac{s-3}{2}\right) \tag{52}$$

with multiple exact roots ($\gamma \neq 1$), $\xi_1 = -2, \xi_2 = 3$. The initial guessed values have been taken as,

$s_1^{(0)} = -1, s_2^{(0)} = 4$. For distinct roots ($\gamma$ =1),

$f_8(s) = sinh\left(\frac{x+2}{2}\right) sinh\left(\frac{x-3}{2}\right)$.

Table 6

Simultaneous determination of all roots

$f_7(s) = e^{s(s-1)^2(s-2)^2(s-3)^3}$

$\xi_1 = 0, \xi_2 = 1, \xi_3 = 2, \xi_4 = 2$

$s_1^{(0)} = 0.1, s_2^{(0)} = 0.9, s_3^{(0)} = 1.8, s_4^{(0)} = 2.9$

Method	CO	CPU	γ	n	e_1	e_2	e_3	e_4
PJ	6	0.203	$\gamma = 1$	5	0.5	0.008	0.002	0.01
PJ	6	0.422	$\gamma \neq 1$	5	0.05	0.003	1.9e-7	8.1e-3
M1-M3	6	0.094	$\gamma = 1$	5	1.0e-5	1.6e-4	0.3e-3	2.1e-4
M1-M3	6	0.063	$\gamma \neq 1$	5	1.3e-4	1.3e-3	2.5e-9	1.9e-5

Table 7

Comparison of single roots finding methods

$f_7(s) = e^{s(s-1)(s-2)(s-3)}$, $s_0 = 0.0, \alpha = 0.1$,CO=4

Method	$\left\|s_i^{(6)} - \alpha\right\|$	$\left\|f_i^{(6)}(s_i)\right\|$	CPU	ρ
MS1	0.01	0.8e-3	0.160	3.0
MS2	0.008	1.5e-7	0.075	3.01
MS3	0.02	1.8e-30	0.290	3.32
KM	0.1	3.0e-2	0.759	1.3
CM	0.02	4.1e-16	0.606	2.5
JM	0.05	6.7e-11	1.982	2.9

Table 8

Simultaneous determination of all roots

$f_8(s) = sinh^5\left(\frac{s+2}{2}\right) sinh^6\left(\frac{s-3}{2}\right)$

$s_1^{(0)} = -1, s_2^{(0)} = 4$

$\xi_1 = -2, \xi_2 = 3$

Method	CO	CPU	γ	n	e_1	e_2
PJ	6	0.203	$\gamma = 1$	5	2.2e-9	1.5e-8
PJ	6	0.422	$\gamma \neq 1$	5	9.8e-8	9.9e-7
M1-M3	6	0.094	$\gamma = 1$	5	2.1e-9	1.4e-10
M1-M3	6	0.063	$\gamma \neq 1$	5	5.5e-11	1.9e-9

Table 9

Comparison of single roots finding methods

$f_8(s) = sinh\left(\frac{x+2}{2}\right) sinh\left(\frac{x-3}{2}\right)$, $s_0 = 2.5, \alpha = 0.1$,CO=4

Method	$\left\|s_i^{(6)} - \alpha\right\|$	$\left\|f_i^{(6)}(s_i)\right\|$	CPU	ρ
MS1	4.3e-741	4.9e-2963	0.031	4.01
MS2	4.4e-804	2.2e-3215	0.036	4.32
MS3	2.1e-656	2.8e-2624	0.041	4.21
KM	1.2e-271	3.5e-1084	0.069	3.91
CM	3.7e-727	8.2e-2907	0.606	4.01
JM	3.2e-500	6.4e-1999	1.531	4.12

4. Conclusion

We have developed here a family of optimal fourth order two-step single root finding methods. We, then derived concrete three fourth order iterative methods for single root finding methods, MS1-MS3 and three simultaneous methods M1-M3 of order six for finding all distinct as well as multiple roots of nonlinear equation. From Tables 2-9 and Figures 1, 2, 3, 4, 5 and 6, we observe that our single root finding methods as well as simultaneous methods (MS1-MS3, M1-M3, respectively) are superior in terms of efficiency, Stability, CPU time and residual errors as compared to the single root finding methods KM, CM, JM and simultaneous M.S. Petković, et al (PJ) [8] method respectively. Higher order single root finding as well as all root finding simultaneous methods ca be developed adopting the similar ways.

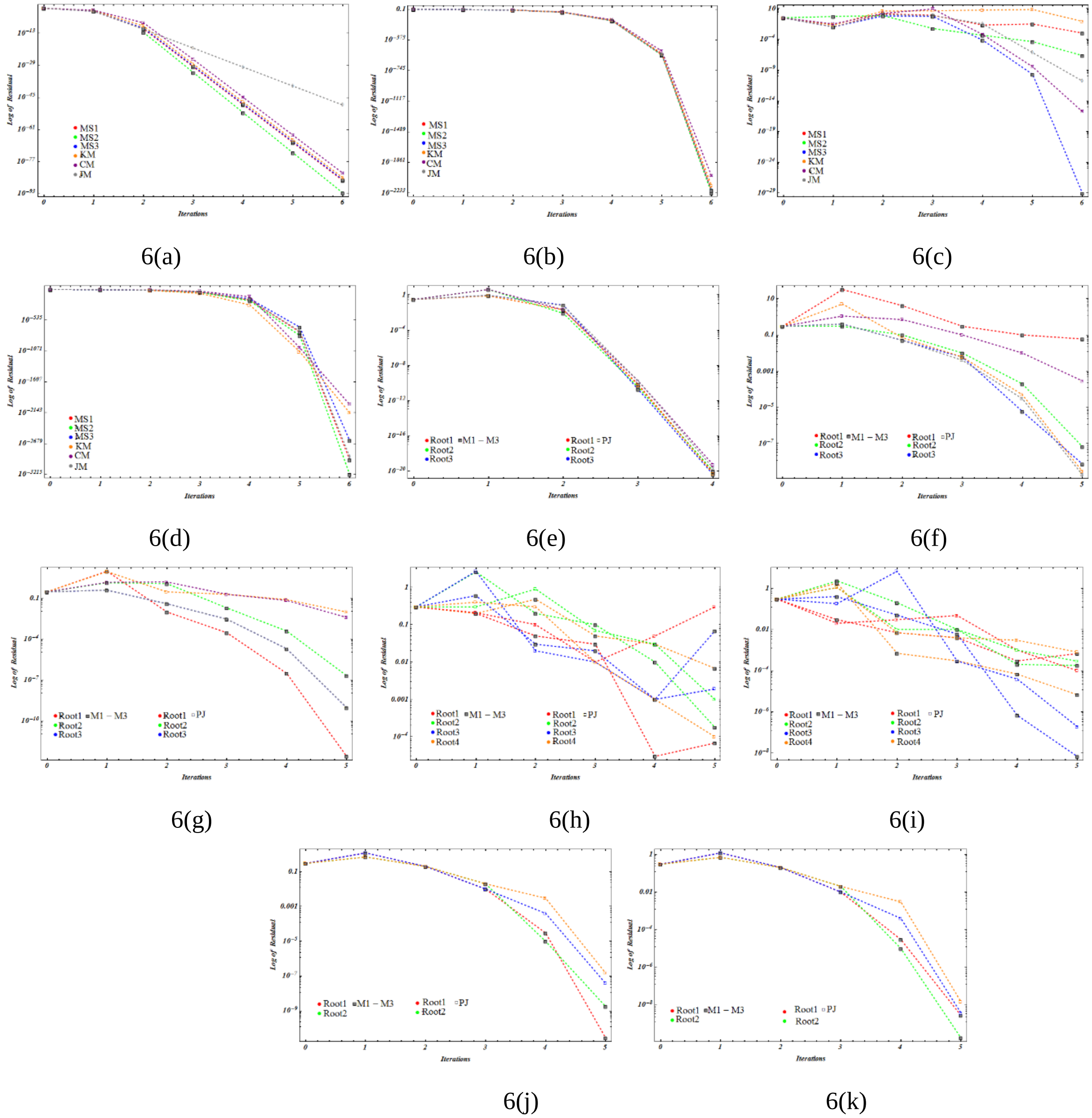


Fig. 6. (a-d), shows residual fall of iterative methods MS1-MS3, KM, CM, JM, (e, g, i, k) presents the residual fall of simultaneous iterative methods (MM1-MM3,PJ6) for finding all distinct roots while, (f, h, j) are for all multiple roots of non-linear function f_5(s), f_6 (s), f_7(s) and f_8(s) respectively

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