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The $(2n+1)^2$-Point Scheme Based on Bivariate Quartic Polynomial

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ABSTRACT

We are going to implement least squares approach to fit the bivariate quartic polynomial to $(2n+1)^2$-perceptions/data, where $n\geq 2$. By taking different values of n, $(2n+1)^2$-point approximating subdivision schemes are built. The proposed scheme can be applied for illustrate individual items as a part of 3D (Three Dimensional) space. The proposed scheme is based on fitting the local least squares bivariate quartic polynomial of degree four to the $(2n+1)^2$-observations. The influence of the proposed scheme is shown by 2D example and its working is presented with the help of different quadrilateral meshes. Subdivision and topological rules are also explained with graphical and mathematical representation. Applications and visual exhibitions of the plan have additionally been displayed to show the implementation of the plan.

Key Words: Approximating Scheme, Least Squares, Quartic Polynomial, 3D Modeling.

1. INTRODUCTION

CAGD (Computer Aided Geometric Design) is the study of geometric design in various fields and since its value is growingrapidly and have an important role in electronic systems, automobiles and airplanes, as well as in many recent engineering applications. A more interesting field in CAGD for designing of curves/surfaces is subdivision. Subdivision is simple to apply and is computationally effective. Only a few of neighboring old points are required for computing new points. This is identical to knot insertion methods found in spline modeling and actually many subdivision methods are clearly generalization of knot insertion.

The concept of using subdivision to develop smooth curves and later, smooth surfaces has been animated for many years. Subdivision provides a compact way to describe geometry with minimum connectivity information. One of the important properties in subdivision is the interpolation/approximation. Improved 4-point ternary approximating subdivision scheme was introduced by Ko et. al. [1]. Bari et. al. [2] worked on shape preserving quaternary subdivision schemes. Bari and Mustafa [3] worked on non-stationary scheme which gives a new era in geometric modeling.

The method of least squares is one of the oldest and by far the most widely used method. Recently, the least

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squares method is frequently used in finding the numerical solutions of the parameters to fit a function to a set of data. Least squares have various forms and its simplest form is called ordinary least squares. A more complicated form of least squares is called weighted least squares. Weighted least squares are better than ordinary least squares due to its property that it can regulate the priority of each observation in the last solution. The new modifications in the least squares method are alternating least squares and partial least squares.

Mustafa and Xuefeng [4] designed volume meshes which exhibited major control over shrinkage of volumetric models. Boye et. al. [5] introduced an alternative approach for curves and surfacesusing subdivision schemes, called Least Squares Subdivision Surfaces. Conti and Dyn [6] worked on BCC-algorithm by subdivision to construct surfaces from initial nets by repeated refinements of nets. Deng and Lin [7] proposed an iterative approximation for least squares fitting (LSPIA), which constructs a series of fitting curves/surfaces by adjusting the control points iteratively. Mustafa et. al. [8] studied l_1-regression and its associated reweighed least squares for data restoration. In addition, they proposed fast numerical optimization method.By fitting local least squares polynomials, Dyn et. al. [9] analyzed univariate, linear, and stationary subdivision schemes for refining noisy data. This was the first attempt to design subdivision schemes for noisy data. In addition, three dimensional objects by subdivision scheme formed by least square are proposed by Mustafa et. al. [10]. Shi et. al. [11], Mustafa and Bari fitting the data to subdivide models by the wide-Ranging families of subdivision schemes. Mustafa et. al. [12] also proposed $(2n)^2$-point subdivision scheme for surface modeling by least squares. Bari [15] also worked on subdivision schemes by least squares approach.

The paper arrangement is given as: Section 2 presenting that how to generate $(2n+1)^2$-point non-tensor product subdivision scheme by bivariate quartic polynomial. Details about the refinement rules are given in Section 3. Some graphical representation is given by the proposed schemes in Section 4. Section 5 contains conclusion and acknowledgement.

We are going to fit the bivariate quartic polynomial to the $(2n+1)^2$-data by using least squares approach.

2. BIVARIATE QUARTIC POLYNOMIAL TO $(2n+1)^2$-OBSERVATIONS BY LEASTSQUARES METHOD

A bivariate quartic polynomial of degree 4 has the following form:

$$p(x,y)=\begin{matrix}\gamma_1+\gamma_2 x+\gamma_3 y+\gamma_4 x^2+\gamma_5 xy+\gamma_6 y^2+\gamma_7 x^3+\gamma_8 x^2 y+\\ \gamma_9 xy^2+\gamma_{10} y^3+\gamma_{11} x^4+\gamma_{12} x^3 y+\gamma_{13} x^2 y^2+\gamma_{14} xy^3+\gamma_{15} y^4\end{matrix} \quad (1)$$

Fitted the data $x_r = r$, $y_s = s$, where $-n<r$, $s<n$ and $n>2$ inEquation (1), we get a general function of bivariatepolynomial having degree four with respect to the observations ($x_r = r, y_s = {}_s, p_{r,s}$) which is given below:

$$p_{r,s}\approx p(r,s)=\begin{matrix}\gamma_1+\gamma_2 x_r+\gamma_3 y_s+\gamma_4 x_r^2+\gamma_5 x_r y_s+\gamma_6 y_s^2+\gamma_7 x_r^3+\gamma_8 x_r^2 y_s+\\ \gamma_9 x_r y_s^2+\gamma_{10} y_s^3+\gamma_{11} x_r^4+\gamma_{12} x_r^3 y_s+\gamma_{13} x_r^2 y_s^2+\gamma_{14} x_r y_s^3+\gamma_{15} y_s^4\end{matrix}$$

To minimizes R, the sum of the squares of differences between $p(r,s)$, (observed value) and $p_{r,s}$(exact value). Differentiating Equation (2):

$$R=\sum_{r=-n}^{n}\sum_{s=-n}^{n}\Big[P_{r,s}-\big(\gamma_1+\gamma_2 x_r+\gamma_3 y_s+\gamma_4 x_r^2+\gamma_5 x_r y_s+\gamma_6 y_s^2+\gamma_7 x_r^3+\gamma_8 x_r^2 y_s+ \\ \gamma_9 x_r y_s^2+\gamma_{10} y_s^3+\gamma_{11} x_r^4+\gamma_{12} x_r^3 y_s+\gamma_{13} x_r^2 y_s^2+\gamma_{14} x_r y_s^2+\gamma_{15} y_s^4\big)\Big]^2 \quad (2)$$

With respect to $\gamma_1, \gamma_2, \ldots \gamma_{15}$ setting them equal to zero, we get fifteen normal Equations and after simplifying those equations, values of unknowns are obtained shown in Equation (1):

$$\gamma_1=\sum_{r=-n}^{n}\sum_{s=-n}^{n}\frac{9}{4\left(4n^2+4n+1\right)\left(16n^4+32n^3-8n^2-24n+9\right)\left(4n^2+4n-15\right)}\big(216n^6-600r^2n^4-600s^2n^4+ \\ 420n^2r^4+400n^2r^2s^2+420n^2n^4+648n^5-1200n^3r^2- \\ 1200n^3s^2+420nr^4+400nr^2s^2+420ns^4-186n^4+ \\ 950n^2r^2+950n^2s^2-315r^4-1500r^2s^2-315s^4- \\ 1452n^3+1550nr^2+1550ns^2-350n^2-525r^2-525s^2+484n-60\big)\,p_{r,s}$$

$$\gamma_3 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-5}{\left(4n^3+8n^2+5n+1\right)\left(4n^4+8n^3-7n^2-11n+6\right)n} \left(-18sn^4+9r^2sn^2+21s^3n^2-36n^3s+9r^2sn+21s^3n+3n^2s-18r^2s-7s^3+21ns-5s\right) p_{r,s}$$

$$\gamma_4 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-45}{4n\left(16n^5+48n^4+40n^3-11n-3\right)\left(16n^6+48n^5-56n^4-192n^3-85n^2+198n-90\right)} \left(120n^8-864r^2n^6-80s^2n^6+840r^4n^4+240r^2s^2n^4+480n^7-2592r^2n^5-240s^2n^5+1680r^4n^3+480r^2s^2n^3+170n^6+24r^2n^4+220s^2n^4-490r^4n^2-1140r^2s^2n^2-1170n^5+4368r^2n^3+840s^2n^3-1330r^4n-1380r^2s^2n-805n^4-140r^2n^2-140s^2n^2+525r^4+1800r^2s^2+900n^3-2756r^2n-600s^2n+515n^2+735r^2-210n\right) p_{r,s}$$

$$\gamma_5 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-3}{n^2\left(4n^4+12n^3+13n^2+6n+1\right)\left(4n^4+8n^3-7n^2-11n+6\right)n} \left(-138n^4rs+105r^3sn^2+105rs^3n^2-276n^3rs+105r^3sn+105rs^3n-21n^2rs-35r^3s-35rs^3+117nrs-32rs\right) p_{r,s}$$

$$\gamma_6 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-45}{4n\left(16n^5+48n^4+40n^3-11n-3\right)\left(16n^6+48n^5-56n^4-192n^3+85n^2+189n-90\right)} \left(120n^8-80r^2n^6-864s^2n^6+840s^4n^4+240r^2s^2n^4+480n^7-2592s^2n^5-240r^2n^5+1680s^4n^3+480r^2s^2n^3+170n^6+24s^2n^4+220r^2n^4-490s^4n^2-1140r^2s^2n^2-1170n^5+4368s^2n^3+840r^2n^3-1330r^2n-1380r^2s^2n-805n^4-140r^2n^2-140s^2n^2+525s^4+1800r^2s^2+900n^3-2756s^2n-600r^2n+515n^2+735s^2-210n\right) p_{r,s}$$

$$\gamma_7 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{35}{n\left(16n^7+64n^6+56n^5-56n^4-91n^3-14n^2+19n+6\right)} \left(-3rn^2+5r^3-3rn+r\right) p_{r,s}$$

$$\gamma_8 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{45}{n^2\left(16n^6+64n^5+88n^4+40n^3-11n^2-14n-3\right)} \left(-n^2s+3r^2s-ns\right) p_{r,s}$$

$$\gamma_9 = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{45}{n^2\left(16n^6+64n^5+88n^4+40n^3-11n^2-14n-3\right)} \left(-rn^2+3rs^2-nr\right) p_{r,s}$$

$$\gamma_{10} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-35}{n\left(16n^7+64n^6+56n^5-56n^4-91n^3-14n^2+19n+6\right)} \left(3n^2s-5s^3+3ns-s\right) p_{r,s}$$

$$\gamma_{11} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-315}{4n\left(64n^9+320n^8+240n^7-960n^6-1428n^5+420n^4+1385n^3+310n^2-261n-90\right)} \left(3n^4-30r^2n^2+35r^4+6n^3-30r^2n-3n^2+25r^2-6n\right) p_{r,s}$$

$$\gamma_{12} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{105}{n^2\left(16n^8+80n^7+120n^6-147n^4-105n^3+5n^2+25n+6\right)} \left(-3n^2rs+5r^3s-3nrs+rs\right) p_{r,s}$$

$$\gamma_{13} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{225}{n^2\left(64n^8+320n^7+560n^6+320n^5-148n^4-220n^3-35n^2+30n+9\right)} \left(n^4-3r^2n^2-3s^2n^2+9r^2s^2+2n^3-3r^2n-3s^2n+n^2\right) p_{r,s}$$

$$\gamma_{14} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{-105}{n^2\left(16n^8+80n^7+120n^6-147n^4-105n^3+5n^2+25n+6\right)} \left(3n^2rs-5rs^3+3nrs-rs\right) p_{r,s}$$

$$\gamma_{15} = \sum_{r=-n}^{n} \sum_{s=-n}^{n} \frac{315}{4n\left(64n^9+320n^8+240n^7-960n^6-1428n^5+420n^4+1385n^3+310n^2-261n-90\right)} \left(3n^4-30s^2n^2+35s^4+6n^3-30s^2n-3n^2+25s^2-6n\right) p_{r,s}$$

By putting the values of γ_i's, i= 1,2,3,…,15 in Equation (1), we have

$$p(x,y) == \frac{1}{\varphi_n} \sum_{r=-n}^{n} \sum_{s=-n}^{n} \left(k_1+k_2x+k_3y+k_4x^2+k_5xy+k_6y^2+k_7x^3+k_8x^2y+k_9xy^2+k_{10}y^3+k_{11}x^4+k_{12}x^3y+k_{13}x^2y^2+k_{14}xy^3+k_{15}y^4\right) p_{r,s} \tag{3}$$

where

$$\varphi_n = 4n^2(n-1)(n+2)(2n-3)(2n+5)(2n+3)^2(2n-1)^2(2n+1)^2(n+1)^2$$

$k_1 =$ $1944n^{12} - 5400n^{10}r^2 - 5400n^{10}s^2 + 3780n^8r^4 + 3600n^8r^2s^2 + 3780n^8s^4 + 11664n^{11} - 2700n^9r^2 - 2700n^9s^2 + 15120n^7r^4 + 1440n^7r^2s^2 + 15120n^7s^4 + 17766n^{10} - 29250n^8r^2 - 29250n^8s^2 + 12285n^6r^4 + 900n^6s^2r^2 + 12285n^6s^4 - 18090n^9 + 4500n^7r^2 + 4500n^7s^2 - 16065n^5r^4 - 47700n^5r^2s^2 - 16065n^5s^4 - 65412n^8 + 88875n^6r^2 + 88875n^6s^2 - 21735n^4r^2 + 25000n^4r^2s^2 - 21735n^4s^4 - 24804n^7 - 4275n^5r^2 - 4275n^5s^2 + 945n^3r^4 - 33300n^3r^2s^2 - 21735n^4r^4 - 31500n^4r^2s^2 - 63675n^4s^2 + 5670n^2r^4 + 27000n^2r^2s^2 + 5670n^2s^4 + 38322n^5 - 13725n^3r^2 - 13725n^3s^2 - 7308n^4 + 9450n^2r^2 + 9450n^2s^2 - 7092n^3 + 1080n^2$

$k_2 =$ $5760n^{10}r - 6720n^8r^3 - 2880n^8rs^2 + 28800n^9r - 26880n^7r^3 - 11520n^7rs^2 + 24960n^8r - 7840n^6r^3 + 1440n^6rs^2 - 72960n^7r + 70560n^5r^3 + 44640n^5rs^2 - 119000n^6r + 61740n^4r^3 + 19260n^4rs^2 + 19320n^5r - 25480n^3r^3 - 49320n^3rs^2 + 85380n^4r - 22680n^2r^3 - 17820n^2rs^2 - 4160n^3r + 6300nr^3 + 16200nrs^2 - 21600n^2r + 4500nr$

$k_3 =$ $5760n^{10}s - 2880n^8r^2s - 6720n^8s^3 + 28800n^9s - 11520n^7r^2s - 26880n^7s^3 + 24960n^8s + 1440n^6r^2s - 7840n^6s^3 - 72960n^7s + 44640n^5r^2s + 70560n^5s^3 - 119000n^6s + 19260n^4r^2s + 61740n^4s^3 + 19320n^5s - 49320n^3r^2s - 25480n^3s^3 + 85380n^4s - 17820n^2r^2s - 22680n^2s^3 - 4160n^3s + 16200nr^2s + 6300ns^3 - 21600n^2s + 4500ns$

$k_4 =$ $-5400n^{10} + 38880n^8r^2 + 3600n^8s^2 - 37800n^6r^4 - 10800n^6r^2s^2 - 27000n^9 + 155520n^7r^2 + 14400n^7s^2 - 113400n^5r^4 - 32400n^5r^2s^2 - 29250n^8 + 115560n^6r^2 + 900n^6s^2 - 53550n^4r^4 + 29700n^4r^2s^2 + 45000n^7 - 197640n^5r^2 - 47700n^5s^2 + 81900n^3r^4 + 113400n^3r^2s^2 + 88875n^6 - 190260n^4r^2 - 31500n^4s^2 + 36225n^2r^4 - 18900n^2r^2s^2 - 4275n^5 + 130320n^3r^2 + 33300n^3s^2 - 23625nr^4 - 81000nr^2s^2 - 63675n^4 + 90945n^2r^2 + 27000n^2s^2 - 13725n^3 - 33075nr^2 + 9450n^2$

$k_5 =$ $2649n^8rs - 20160n^6r^3s - 20160n^6rs^3 + 105984n^7rs - 60480n^5r^3s - 60480n^5rs^3 + 17280n^6rs + 36960n^4r^3s + 36960n^4rs^3 - 319104n^5rs + 174720n^3r^3s + 174720n^3rs^3 - 216840n^4rs + 10500n^2r^3s + 10500n^2rs^3 + 221808n^3rs - 86940nr^3s - 86940nrs^3 + 90924n^2rs + 18900r^3s + 18900rs^3 - 90828nrs + 1780rs$

$k_6 =$ $5400n^{10} + 3600n^8r^2 + 38800n^8s^2 - 10800n^6r^2s^2 - 37800n^6s^4 - 27000n^9 + 14400n^7r^2 + 155520n^7s^2 - 32400n^5r^2s^2 - 113400n^4r^4 - 29250n^8 + 900n^6r^2 + 115560n^6s^2 + 29700n^4r^4s^2 - 5355n^4s^2 + 45000n^7 - 47700n^5r^2 - 97640n^5s^2 + 81900n^3s^4 + 113400n^3r^2s^2 + 88875n^6 - 31500n^4r^2 - 190260n^4s^2 - 18900n^2r^2s^2 + 36225n^2r^4 - 4275n^5 + 33300n^3r^2 + 130320n^3s^2 - 81000nr^2s^2 - 23625ns^4 - 63675n^4+27000n^2r^2 + 90945n^2s^2 - 13725n^3 - 33075ns^2 + 9450n^2$

$k_7 =$ $- 6720n^8r + 11200n^6r^3 - 26880n^7r + 33600n^5r^3 - 7840n^6r - 16800n^4r^3 + 70560n^5r - 89600n^3r^3 + 61740n^4r - 18900n^2r^3 - 25480n^3r + 31500nr^3 - 22680n^2r + 6300nr$

$k_8 =$ $-2880n^8s + 8640n^6r^2s - 11520n^7s + 25920n^5r^2s + 1440n^6s - 30240n^4r^2s + 44640n^5s - 103680n^3r^2s + 19260n^4s + 45900n^2r^2s - 49320n^3s + 102060nr^2s - 17820n^2s - 48600r^2s + 16200ns$

$k_9 =$ $-2880n^8r + 8640n^6rs^2 - 11520n^7r + 25920n^5rs^2 + 1440n^6r - 30240n^4rs^2 + 44640n^5r - 103680n^3rs^2 + 19260n^4r + 45900n^2rs^2 - 49320n^3r - 102060nrs^2 - 17820n^2r - 48600rs^2 + 16200nr$

$k_{10} =$ $-6720n^8s + 11200n^6s^3 - 26800n^7s + 33600n^5s^3 - 7840n^6s - 16800n^4s^3 + 70560n^5s - 89600n^3s^3 + 16740n^4s - 18900n^2s^3 - 25480n^3s + 31500ns^3 - 22680n^2s + 6300ns$

$k_{11} =$ $3780n^8 - 37800n^6r^2 + 44100n^4r^4 + 15120n^7 - 113400n^5r^2 + 88200n^3r^4 + 12285n^6 - 53550n^4r^4r^2 + 11025n^2r^4 - 16065n^5 + 81900n^3r^2 - 33075nr^4 - 21735n^4 + 36225n^2r^2 + 945n^3 - 23625nr^2 + 5670n^2$

$k_{12} =$ $- 21160n^6rs + 33600n^4r^3s - 60480n^5rs + 67200n^3r^3s + 36960n^4rs - 117600n^2r^3s + 174720n^3rs - 151200nr^3s + 10500n^2rs + 94500r^3s - 86nrs + 18900rs$

$k_{13} =$ $3600n^8 - 10800n^6r^2 - 10800n^6s^2 + 32400n^4r^2s^2 + 14400n^7 - 32400n^5r^2 - 32400n^5s^2 + 64800n^3r^2s^2 + 900n^6 + 29700n^4r^2 + 29700n^4s^2 - 153900n^2r^2s^2 - 47700n^5 + 113400n^3r^2 + 113400n^3s^2 - 186300nr^2s^2 - 31500n^4 - 18900n^2r^2 - 18900n^2s^2 + 243000r^2s^2 + 33300n^3 - 81000nr^2 - 81000ns^2 + 27000n^2$

$k_{14} =$ $-20160n^6rs + 33600n^4rs^3 - 60480n^5rs + 67200n^3rs^3 + 36960n^4rs - 117600n^2rs^3 + 174720n^3rs - 151200nrs^3 + 10500n^2rs + 94500rs^3 - 86940nrs + 18900rs$

$k_{15} =$ $3780n^8 - 37800n^6s^2 + 44100n^4s^4 + 15120n^7 - 113400n^5s^2 + 88200n^3s^4 + 12285n^6 - 53550n^4s^2 + 11025n^2s^4 - 16065n^5 + 81900n^3s^2 - 33075ns^4 - 21735n^4 + 36225n^2s^2 + 945n^3 - 23625ns^2 + 5670n^2$

So by least squares method, the bivariate quartic polynomial (1) is the distinctivepolynomial to fit the data points in 3D space. To fit the data for the modeling of discrete objects, we are going to present $(2n+1)^2$-point subdivision scheme.

2.1 Surface Subdivision Scheme

In this section, we present $(2n+1)^2$-point approximating subdivision scheme to fit the data for the modeling of different things.

Evaluating Equation (3) at the given points $\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{3}{4},\frac{1}{4}\right),\left(\frac{1}{4},\frac{3}{4}\right)$ and $\left(\frac{3}{4},\frac{3}{4}\right)$

We get following equations

$$p\left(\frac{1}{4},\frac{1}{4}\right)=\frac{1}{\phi_n}\sum_{r=-n}^{n}\sum_{s=-n}^{n}\left(k_1+\frac{1}{4}k_2+\frac{1}{4}k_3+\frac{1}{16}k_4+\frac{1}{16}k_5+\frac{1}{16}k_6+\frac{1}{64}k_7+\frac{1}{64}k_8+\frac{1}{64}k_9+\frac{1}{64}k_{10}+\frac{1}{256}k_{11}+\frac{1}{256}k_{12}+\frac{1}{256}k_{13}+\frac{1}{256}k_{14}+\frac{1}{256}k_{15}\right)p_{r,s} \tag{4}$$

$$p\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{\phi_n}\sum_{r=-n}^{n}\sum_{s=-n}^{n}\left(k_1+\frac{3}{4}k_2+\frac{1}{4}k_3+\frac{9}{16}k_4+\frac{3}{16}k_5+\frac{1}{16}k_6+\frac{27}{64}k_7+\frac{9}{64}k_8+\frac{3}{64}k_9+\frac{1}{64}k_{10}+\frac{81}{256}k_{11}+\frac{27}{256}k_{12}+\frac{9}{256}k_{13}+\frac{3}{256}k_{14}+\frac{1}{256}k_{15}\right)p_{r,s} \tag{5}$$

$$p\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{\phi_n}\sum_{r=-n}^{n}\sum_{s=-n}^{n}\left(k_1+\frac{1}{4}k_2+\frac{3}{4}k_3+\frac{1}{16}k_4+\frac{3}{16}k_5+\frac{9}{16}k_6+\frac{1}{64}k_7+\frac{3}{64}k_8+\frac{9}{64}k_9+\frac{27}{64}k_{10}+\frac{1}{256}k_{11}+\frac{3}{256}k_{12}+\frac{9}{256}k_{13}+\frac{27}{256}k_{14}+\frac{81}{256}k_{15}\right)p_{r,s} \tag{6}$$

$$p\left(\frac{3}{4},\frac{3}{4}\right)=\frac{1}{\phi_n}\sum_{r=-n}^{n}\sum_{s=-n}^{n}\left(k_1+\frac{3}{4}k_2+\frac{3}{4}k_3+\frac{9}{16}k_4+\frac{9}{16}k_5+\frac{9}{16}k_6+\frac{27}{64}k_7+\frac{27}{64}k_8+\frac{27}{64}k_9+\frac{27}{64}k_{10}+\frac{81}{256}k_{11}+\frac{81}{256}k_{12}+\frac{81}{256}k_{13}+\frac{81}{256}k_{14}+\frac{81}{256}k_{15}\right)p_{r,s} \tag{7}$$

By substituting the values of k_i, s, $i = 1,2,3,\ldots,15$ and φ_n and in Equations (4-7), evaluating at n=2 and changing the notations we have

$$p\left(\frac{1}{4},\frac{1}{4}\right)=p^{k+1}_{2i-1,2j-1},\ p\left(\frac{3}{4},\frac{1}{4}\right)=p^{k+1}_{2i,2j-1},$$
$$p\left(\frac{1}{4},\frac{3}{4}\right)=p^{k+1}_{2i-1,2j},\ p\left(\frac{3}{4},\frac{3}{4}\right)=p^{k+1}_{2i,2j},\ p_{r,s}=p^{k}_{i+r,j+s}$$

We get following 25-point subdivision scheme with 4 rules for surface modeling as well as for data fitting:

$$
\begin{aligned}
p_{2i-1,2j-1}^{k+1} = {} & \frac{83633}{1254400} p_{i-2,j-2}^{k} - \frac{116699}{1254400} p_{i-2,j-1}^{k} + \frac{144353}{250880} p_{i-2,j}^{k} - \frac{33979}{627200} \\
& p_{i-2,j+1}^{k} + \frac{10057}{250880} p_{i-2,j+2}^{k} - \frac{116699}{1254400} p_{i-1,j-2}^{k} + \frac{189849}{1254400} p_{i-1,j-2}^{k} - \frac{17413}{250880} \\
& p_{i-1,j-1}^{k} + \frac{189849}{1254400} p_{i-1,j}^{k} - \frac{15077}{1254400} p_{i-1,j+1}^{k} - \frac{62679}{627200} p_{i-1,j+2}^{k} - \frac{144353}{2508800} \\
& p_{i,j-2}^{k} + \frac{189849}{1254400} p_{i,j-1}^{k} + \frac{254487}{627200} p_{i,j}^{k} + \frac{289529}{1254400} p_{i,j+1}^{k} + \frac{194193}{2508800} p_{i,j+2}^{k} - \frac{33979}{627200} \\
& p_{i+1,j-2}^{k} - \frac{15077}{1254400} p_{i+1,j-1}^{k} + \frac{289529}{1254400} p_{i+1,j}^{k} + \frac{18987}{250880} p_{i+1,j+1}^{k} - \frac{44179}{1254400} p_{i+1,j+2}^{k} + \\
& \frac{10057}{250880} p_{i+2,j-2}^{k} - \frac{62679}{627200} p_{i+2,j-1}^{k} + \frac{194193}{2508800} p_{i+2,j}^{k} - \frac{44179}{1254400} p_{i+2,j+1}^{k} - \frac{5407}{1254400} p_{i+2,j+2}^{k}
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
p_{2i,2j-1}^{k+1} = {} & \frac{79861}{1254400} p_{i-2,j-2}^{k} - \frac{1067}{12544} p_{i-2,j-1}^{k} + \frac{124513}{2508800} p_{i-2,j}^{k} - \frac{87877}{1254400} \\
& p_{i-2,j+1}^{k} + \frac{73897}{1254400} p_{i-2,j+2}^{k} - \frac{107}{1960} p_{i-1,j-2}^{k} - \frac{79853}{1254400} p_{i-1,j-1}^{k} + \frac{164009}{1254400} \\
& p_{i-1,j}^{k} - \frac{44209}{1254400} p_{i-1,j+1}^{k} - \frac{106217}{1254400} p_{i-1,j+2}^{k} - \frac{71487}{2508800} p_{i,j-2}^{k} + \frac{42489}{1254400} \\
& p_{i,j-1}^{k} + \frac{173487}{627200} p_{i,j}^{k} + \frac{137689}{1254400} p_{i,j+1}^{k} - \frac{39567}{2508800} p_{i,j+2}^{k} + \frac{54223}{1254400} p_{i+1,j-2}^{k} + \frac{108671}{1254400} \\
& p_{i+1,j-1}^{k} + \frac{431689}{1254400} p_{i+1,j}^{k} + \frac{250547}{1254400} p_{i+1,j+1}^{k} + \frac{2453}{31360} p_{i+1,j+2}^{k} - \frac{8423}{1254400} \\
& p_{i+2,j-2}^{k} - \frac{118957}{1254400} p_{i+2,j-1}^{k} + \frac{305393}{2508800} p_{i+2,j}^{k} + \frac{11}{12544} p_{i+2,j+1}^{k} - \frac{73579}{1254400} p_{i+2,j+2}^{k}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
p_{2i-1,2j}^{k+1} = {} & \frac{79861}{1254400} p_{i-2,j-2}^{k} - \frac{107}{1960} p_{i-2,j-1}^{k} - \frac{71487}{2508800} p_{i-2,j}^{k} + \frac{54223}{1254400} \\
& p_{i-2,j+1}^{k} - \frac{8423}{1254400} p_{i-2,j+2}^{k} - \frac{1067}{1254400} p_{i-1,j-2}^{k} - \frac{79853}{1254400} p_{i-1,j-1}^{k} + \frac{42489}{1254400} \\
& p_{i-1,j}^{k} + \frac{108671}{1254400} p_{i-1,j+1}^{k} - \frac{118957}{1254400} p_{i-1,j+2}^{k} + \frac{124513}{2508800} p_{i,j-2}^{k} + \frac{164009}{1254400} \\
& p_{i,j-1}^{k} + \frac{173487}{627200} p_{i,j}^{k} + \frac{431689}{1254400} p_{i,j+1}^{k} + \frac{305393}{2508800} p_{i,j+2}^{k} - \frac{87877}{1254400} p_{i+1,j-2}^{k} - \frac{44209}{1254400} \\
& p_{i+1,j-1}^{k} + \frac{137689}{1254400} p_{i+1,j}^{k} + \frac{250547}{1254400} p_{i+1,j+1}^{k} + \frac{11}{12544} p_{i+1,j+2}^{k} + \frac{73897}{1254400} \\
& p_{i+2,j-2}^{k} - \frac{106217}{1254400} p_{i+2,j-1}^{k} - \frac{39567}{2508800} p_{i+2,j}^{k} + \frac{2453}{31360} p_{i+2,j+1}^{k} - \frac{73579}{1254400} p_{i+2,j+2}^{k}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
p_{2i,2j}^{k+1} = {} & \frac{10013}{250880} p_{i-2,j-2}^{k} - \frac{35123}{1254400} p_{i-2,j-1}^{k} - \frac{86207}{2508800} p_{i-2,j}^{k} + \frac{6753}{627200} p_{i-2,j+1}^{k} + \frac{36093}{1254400} \\
& p_{i-2,j+2}^{k} - \frac{35123}{1254400} p_{i-1,j-2}^{k} - \frac{36377}{1254400} p_{i-1,j-1}^{k} + \frac{24329}{1254400} p_{i-1,j}^{k} + \frac{10191}{250880} \\
& p_{i-1,j+1}^{k} - \frac{69267}{627200} p_{i-1,j+2}^{k} - \frac{86207}{2508800} p_{i,j-2}^{k} + \frac{24329}{1254400} p_{i,j-1}^{k} + \frac{95687}{627200} \\
& p_{i,j}^{k} + \frac{278569}{1254400} p_{i,j+1}^{k} + \frac{40913}{2508800} p_{i,j+2}^{k} + \frac{6753}{627200} p_{i+1,j-2}^{k} + \frac{10191}{250880} \\
& p_{i+1,j-1}^{k} + \frac{278569}{1254400} p_{i+1,j}^{k} + \frac{433463}{1254400} p_{i+1,j+1}^{k} + \frac{166757}{1254400} p_{i+1,j+2}^{k} + \frac{36093}{1254400} \\
& p_{i+2,j-2}^{k} - \frac{69267}{627200} p_{i+2,j-1}^{k} + \frac{40913}{2508800} p_{i+2,j}^{k} + \frac{166757}{1254400} p_{i+2,j+1}^{k} - \frac{26387}{2508800} p_{i+2,j+2}^{k}
\end{aligned}
\tag{11}
$$

Here k shows refinement level whereas $p_{i,j}^{k+1}$ and $p_{i,j}^{k}$ represent control points (data points) at level $k+1$ and k respectively.

Remark 3.1. For the values of n=3,4,5,6…in Equations (4-7) with changing the notations, a class of approximating subdivision schemes called $(2n+1)^2$-point scheme is obtained.

3. IMPLEMENTATION OF THE SCHEME

Subdivision step comprises two rules: the topological rule and subdivision rule. The topological rule describes how the refined mesh is created from the original mesh. The subdivision rule computes the points of the new mesh by taking a linear combination of the local $(2n+1)^2$-points of coarse mesh. These steps are carried out for next local $(2n+1)^2$-points of coarse mesh. Both iterative steps are repeated for next level of iteration. These steps continue until the resulting surface/ mesh become sufficiently smooth.

The subdivision rules are explained in Fig. 1 while the topological/connectivity rules are explained in Fig. 2. Solid diamonds are coarse points while solid lines show coarse mesh. Further details are given below.

3.1 First Refined Rule

Geometrical pattern (i.e. stencil) of coarse mesh to determine $p_{2i-1,2j-1}^{k+1}$ (new point) of refined mesh is shown in Fig. 1(a). The weights a_1 to a_{15} in Fig. 1(a) are taken from the rule (4)

$$
\begin{aligned}
& a_1 = \frac{83633}{1254400}, a_2 = -\frac{116699}{1254400}, a_3 = \frac{144353}{2508800}, a_4 = -\frac{33939}{627200}, a_5 = \frac{10057}{250880}, \\
& a_6 = -\frac{17413}{250880}, a_7 = \frac{189849}{1254400}, a_8 = -\frac{15077}{1254400}, a_9 = -\frac{62679}{627200}, a_{10} = \frac{254487}{627200}, \\
& a_{11} = \frac{289529}{1254400}, a_{12} = \frac{194192}{2508800}, a_{13} = \frac{18987}{250880}, a_{14} = -\frac{44179}{1254400}, a_{15} = -\frac{5407}{1254400}
\end{aligned}
$$

3.2 Second Refined Rule

Geometrical pattern of coarse mesh to determine $p_{2i,2j-1}^{k+1}$ (new point) of refined mesh is shown in

Fig. 1(b). The weights b_1 to b_{25} in Fig. 1(b) are taken by the rule defined in Equation (5).

$$b_1=\frac{79861}{1254400},b_2=-\frac{1067}{12544},b_3=\frac{124513}{2508800},b_4=-\frac{87877}{1254400},b_5=\frac{73897}{1254400},$$
$$b_6=-\frac{107}{1960},b_7=-\frac{79853}{1254400},b_8=\frac{164009}{1254400},b_9=-\frac{44209}{1254400},b_{10}=-\frac{106217}{1254400},$$
$$b_{11}=-\frac{71487}{2508800},b_{12}=\frac{42489}{1254400},b_{13}=\frac{173487}{627200},b_{14}=\frac{137689}{1254400},b_{15}=-\frac{39567}{2508800},$$
$$b_{16}=\frac{54223}{1254400},b_{17}=\frac{108671}{1254400},b_{18}=\frac{431689}{1254400},b_{19}=\frac{250547}{1254400},b_{20}=\frac{2453}{31360},$$
$$b_{21}=-\frac{8423}{1254400},b_{22}=-\frac{118957}{1254400},b_{23}=\frac{305393}{2508800},b_{24}=\frac{11}{12544},b_{25}=-\frac{73579}{1254400}$$

3.3 Third Refined Rule

Geometrical pattern of coarse mesh to determine $p_{2i-1,2j}^{k+1}$ (new point) of refinedmesh is shown in Fig. 1(b). The weights b_1 to b_{25} in Fig. 1(b) are taken by the rule defined in Equation (6).

$$b_1=\frac{79861}{1254400},b_2=-\frac{107}{1960},b_3=\frac{-71487}{2508800},b_4=\frac{54223}{1254400},b_5=-\frac{8423}{1254400},$$
$$b_6=-\frac{1067}{12544},b_7=-\frac{79853}{1254400},b_8=\frac{42489}{1254400},b_9=\frac{108671}{1254400},b_{10}=-\frac{118957}{1254400},$$
$$b_{11}=\frac{124513}{2508800},b_{12}=\frac{164009}{1254400},b_{13}=\frac{173487}{627200},b_{14}=\frac{431689}{1254400},b_{15}=\frac{305393}{2508800},$$
$$b_{16}=-\frac{87877}{1254400},b_{17}=-\frac{44209}{1254400},b_{18}=\frac{137689}{1254400},b_{19}=\frac{250547}{1254400},b_{20}=\frac{11}{12544},$$
$$b_{21}=-\frac{73897}{1254400},b_{22}=-\frac{106217}{1254400},b_{23}=-\frac{39567}{2508800},b_{24}=\frac{2453}{31360},b_{25}=-\frac{73579}{1254400}$$

3.4 Fourth Refined Rule

Geometrical pattern of coarse mesh to determine $p_{2i,2j}^{k+1}$ (new point) of refined mesh is shown in Fig. 1(d). The weights c_1 to c_{15} in Fig. 1(d) are taken by the rule defined in Equation (7).

$$c_1=\frac{10013}{250880},c_2=-\frac{35123}{1254400},c_3=-\frac{86207}{2508800},c_4=\frac{6753}{627200},c_5=\frac{36093}{1254400},$$
$$c_6=-\frac{36377}{1254400},c_7=\frac{24329}{1254400},c_8=\frac{10191}{250880},c_9=-\frac{69267}{627200},c_{10}=\frac{95687}{627200},$$
$$c_{11}=\frac{278569}{1254400},c_{12}=\frac{40913}{2508800},c_{13}=\frac{433463}{1254400},c_{14}=\frac{166757}{1254400},c_{15}=-\frac{26387}{250880}$$

3.5 Fifth Refined Rule

Fig. 2(a) shows the four new points $p_{2i-1,2j-1}^{k+1}, p_{2i,2j-1}^{k+1}, p_{2i-1,2j}^{k+1}, p_{2i,2j}^{k+1}$ corresponding to face of coarse mesh. Similarly, corresponding to each face of the coarse mesh four new points are introduced. These points are connected to get the part of refined mesh.

3.6 Sixth Refined Rule

Fig. 2(b) shows both the coarse and refined meshes.

3.7 Seventh Refined Rule

Fig. 2(c) shows only refined mesh. In this rule we omit all coarse points and coarse grid/mesh. These refined rules are repeated again and again until we get smooth mesh also called limit surface.

4. APPLICATIONS OF THE SCHEME

In this section, we are presenting different snapshots to check the efficiency and behavior of the 25-point approximating subdivision scheme.

We consider quadrilateral mesh as an initial mesh as shown in Fig. 1. Then we iteratively refine this mesh by 25-point scheme and finally get a smooth object. The visual details in Fig. 3(a-d).

Fig. 3(a) is an initial quadrilateral mesh. The points are joined by straight lines(edges). A quadrilateral face(regular face) is formed by four edges.

Fig. 3(b) shows the first refinement by 25-point approximating subdivision scheme. One can see that next level is compact than the previous mesh.

Fig. 3(c) shows the second refinement by proposed approximating scheme.

Fig. 3(d) shows the refined mesh after sufficiently large number of iterative steps.

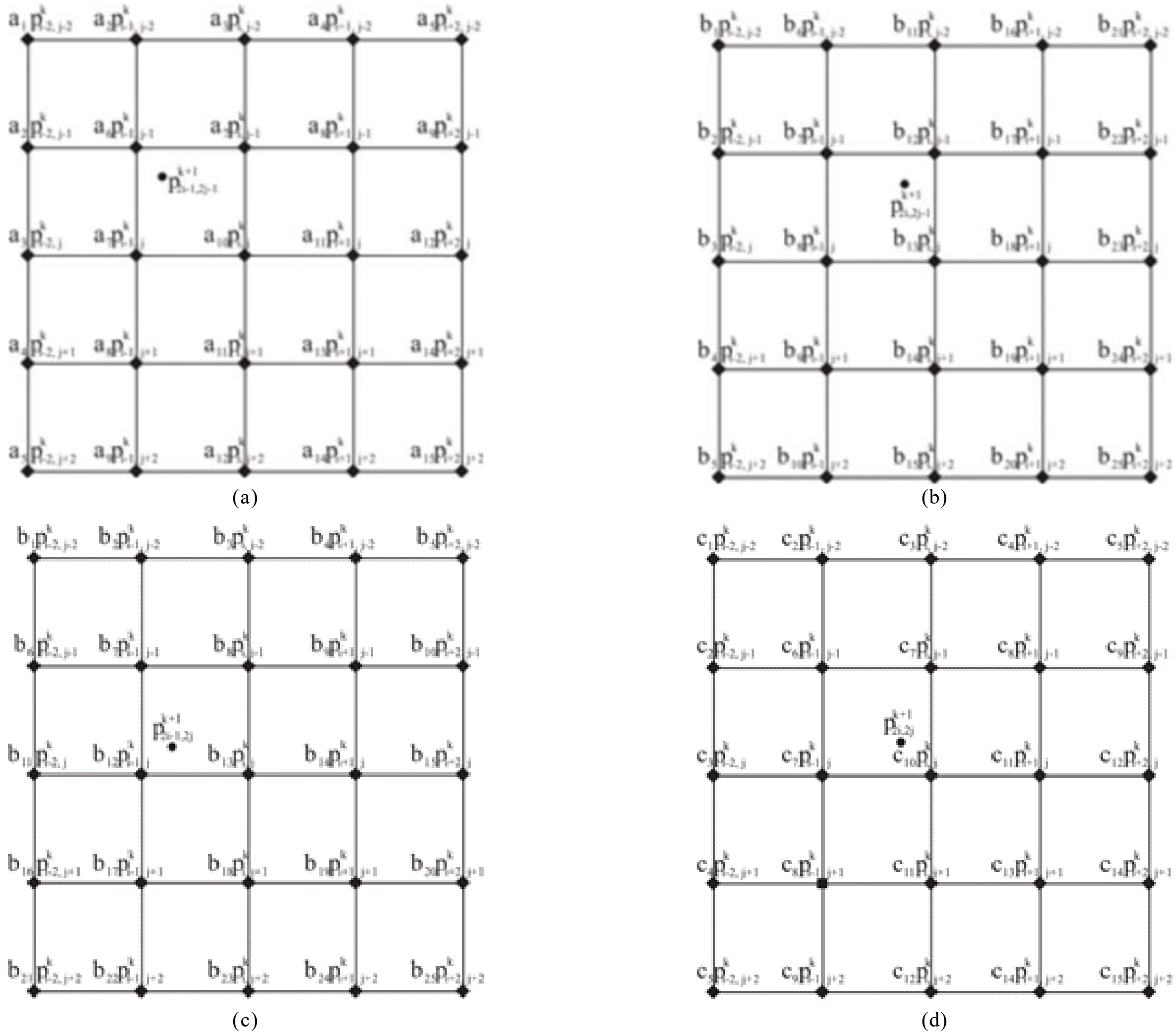


FIG. 1(a-d) SHOW THE GEOMETRICAL REPRESENTATION TO INSERT NEW POINT $p^{k+1}_{2i-1,2j-1}, p^{k+1}_{2i,2j-1}, p^{k+1}_{2i-1,2j}, p^{k+1}_{2i,2j}$ (SOLID CIRCLES) RESPECTIVELY

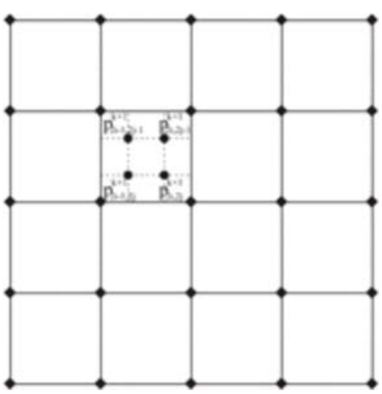


FIG. 2(a). SHOWS FOUR NEW POINTS $p^{k+1}_{2i-1,2j-1}, p^{k+1}_{2i,2j-1}, p^{k+1}_{2i-1,2j}, p^{k+1}_{2i,2j}$ CORRESPONDING TO ONE COARSE FACE OF COARSE MESH

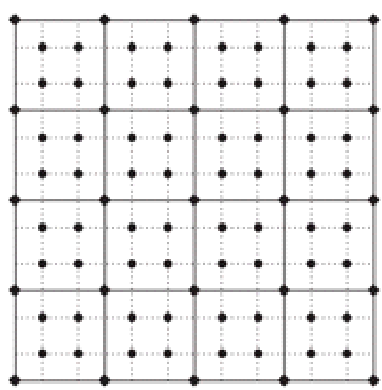


FIG. 2(b). SHOWS COARSE AND REFINED MESHES

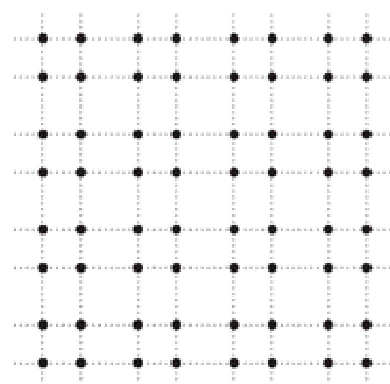
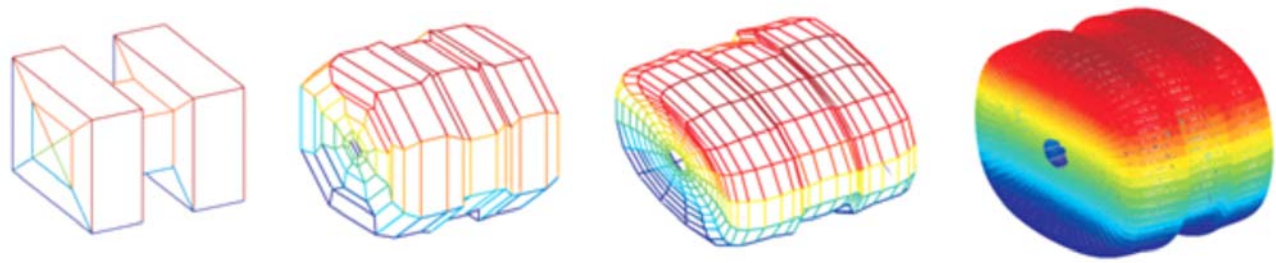


FIG. 2(c). SHOWS REFINED MESH



(a) (b) (c) (d)

FIG. 3(a) SHOWS INITIAL MESH, FIRST, SECOND AND LIMIT SURFACE GENERATED BY 25-POINT SUBDIVISION SCHEME ARE SHOWN IN FIG. 2(b-d) RESPECTIVELY

5. CONCLUSION

Fitting the bivariate quartic polynomial to the information by least squares strategy then from this fitted polynomial, $(2n+1)^2$-point approximating subdivision scheme have been produced for surface displaying. The guidelines for usage of the scheme have been examined in subtle element. A group of schemes can be acquired by taking distinctive estimations of *n*. These schemes can create objects from quadrilateral cross sections.

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