

The Physical Significance of Time Conformal Minkowski Spacetime

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ABSTRACT

The Minkowski spacetime is flat and there is no source of gravitation. The time conformal factor is adding some curvature to this spacetime which introduces some source of gravitation to the spacetime. For the Minkowski spacetime the Einstein Field equation tells nothing, because all the components of the Ricci curvature tensor are zero, but for the time conformal Minkowski spacetime some of them are non zero. Calculating the components of the Ricci tensor and using the Einstein field equations, expressions for the cosmological constant are calculated. These expressions give some information for the cosmological constant. Generally, the Noether symmetry generator corresponding to the energy content in the spacetime disappears by introducing the time conformal factor, but our investigations in this paper reveals that it appears somewhere with some re-scale factor. The appearance of the time like isometry along with some re-scaling factor will re-scale the energy content in the corresponding particular time conformal Minkowski spacetime. A time conformal factor of the form $e^{u(t)}$ is introduced in the Minkowski spacetime for the investigation of the cosmological constant. The Noether symmetry equation is used for the Lagrangian of general time conformal Minkowski spacetime to find all those particular Minkowski spacetimes that admit the time conformal factor. Besides the Noether symmetries the cosmology constant is calculated in the corresponding spacetimes.

Key Words: Time Conformal Minkowski Spacetime, Cosmological Constant, Einstein Field Equations, Noether Symmetry, Conservation Laws.

1. INTRODUCTION

A manifold resembles the Euclidean space locally and can be covered by patches of Euclidean space. However, the global behavior of the manifold is entirely different from the Euclidean geometry. The manifold of Minkowski spacetime plays an important role in both special as well as general relativity. The knowledge of differential geometry and differential manifold is necessary to explain Minkowski spacetime geometry.

In the following, a brief discussion of manifold and causality has been given before discussing the Minkowski metric.

Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$. A collection is defined to be a manifold if every member of M has a continuous open neighborhood which forms a bijective function to an open set of \mathbf{R}^n for some n . Mathematically, [1-2] consider the class \mathcal{C}^r atlas (U_α, Φ_α) of manifold M , where U_α are subsets of the

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manifold M and Φ_α is the corresponding one-one mapping of U_α to open sets in \mathbf{R}^n , then the following conditions are fulfilled for M . U_α covers M , that is $U_\alpha U_\alpha = M$; If $U_\alpha \cap U_\beta \neq \emptyset$, then the mapping $\Phi_\alpha \circ \Phi_\beta^{-1}: \Phi_\beta(U_\alpha \cap U_\beta) \rightarrow \Phi_\alpha(U_\alpha \cap U_\beta)$ is C^r of an open subset of \mathbf{R}^n to an open subset of \mathbf{R}^n

Symmetries play an important role in the solution of the Einstein field equations [3-5]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (1)$$

We used Noether symmetries to classify plane symmetric, cylindrically symmetric and spherically symmetric spacetimes according to their Noether symmetries [6-10]. Plane symmetric and cylindrically symmetric spacetime admitting the time conformal factor is studied in [11-12], while the dynamics of neutral and charge particles in the time conformal Schwarzschild spacetime is studied in [13]. Different types of spacetimes symmetries are given in references [14-16].

Here we consider the time conformal Minkowski spacetime:

$$ds^2 = e^{u(t)}(dt^2 - dx^2 - dy^2 - dz^2) \quad (2)$$

$u(t)$ is a general function of time and the Noether symmetry equation to find particular classes for the spacetime (2). Different values of $u(t)$ give different symmetry structure of spacetime (2). There are seventeen Noether symmetries admitting by the exact Minkowski spacetime:

$$ds^2 = (dt^2 - dx^2 - dy^2 - dz^2) \quad (3)$$

Ten of the seventeen Noether symmetries form the Poincare group, four are the Galilean transformations, one is scaling, one is the conformal transformation, and one is the symmetry corresponding to the Lagrangian of spacetime given in Equation (2). $e^{u(t)}$, the conformal factor, will reduce the Noether symmetries' number for the spacetime (2). Decreasing the number of Noether symmetries mean decreasing conservation laws. The study of particular classes for

particular values of function $u(t)$ will tell which conservation law holds and which one violates. Specifically, the symmetry generator ∂_t will disappear due to the time conformal factor $e^{u(t)}$, which is linked to the energy content in the specified space time. But this symmetry will occur along with some re-scale term, as our investigation in this work confirms it. The symmetries correspond to the boosts and Galilean transformations will also disappear, because of the introduction of the same conformal factor $e^{u(t)}$, which can not be recovered. In this work, the cosmological constant Λ is also calculated in each spacetime. Furthermore, the effect of time on this constant is investigated, too.

2. NOETHER SYMMETRY AND THE CONSERVATION LAWS

The Lagrangian corresponding to Equation (2) is:

$$L = e^{u(t)}(\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2) \quad (4)$$

Using this Lagrangian in the Noether symmetry equation:

$$N^1L + (D\chi)L = DA \quad (5)$$

and obtaining system of Noether symmetry determining partial differential equations. The solution of which will give different particular values for the function $u(t)$ along with Noether symmetries and conservation laws. This solution will consist of all time conformal Minkowski spacetime, the symmetry structure of each class will be discussed correspondingly. Where

$$N^1 = \chi \frac{\partial}{\partial s} + \eta^i \frac{\partial}{\partial x^i} + \eta_s^i \frac{\partial}{\partial x^i}, \quad (6)$$

represents extension of the first order of the Noether symmetry operator:

$$N = \chi \frac{\partial}{\partial s} + \eta^i \frac{\partial}{\partial x^i}, \quad (7)$$

D is the total differential operator of the form:

$$D = \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i}, \quad (8)$$

and A is the gauge function. Using the Lagrangian (4) in Equation (5), a system of nineteen determining PDEs (Partial Differential Equation) are obtained as follows:

$$\begin{aligned} A_s &= 0, & -2u(t)\eta_{3s} - A_y &= 0, & \chi_z &= 0, \\ \chi_y &= 0, & \chi_t &= 0, & \eta_{3z} + \eta_{4y} &= 0, \\ \chi_x &= 0, & \eta_{2z} + \eta_{4x} &= 0, & \chi_x &= 0, \\ \eta_{2y} + \eta_{3x} &= 0, & \eta_{1z} + \eta_{4t} &= 0, \\ u(t)\chi_t &= 0, & \eta_{1y} - \eta_{3t} &= 0, & \eta_{1x} - \eta_{2t} &= 0, \\ \eta_1 u_t + 2(\eta_{1t} - \chi_s)u(t) + u(t)\chi_s &= 0, & & & & (9) \\ -\eta_1 u_t - 2(-\chi_s + \eta_{2x})u(t) - u(t)\chi_s &= 0, & & & & \\ -\eta_1 u_t - 2(-\chi_s + \eta_{3y})u(t) - u(t)\chi_s &= 0, & & & & \\ -\eta_1 u_t - 2(-\chi_s + \eta_{4z})u(t) - u(t)\chi_s &= 0, & & & & \\ -2u(t)\eta_{4s} - A_z &= 0, & 2u(t)\eta_{1s} - A_t &= 0, & & \\ -2u(t)\eta_{2s} - A_x &= 0. & & & & \end{aligned}$$

The solution of the system Equation (9) will give the spacetimes of our interest. This solution will be consists of all the time conformal Minkowski spacetimes. The conservation law corresponding to each Noether symmetry is:

$$\phi = \frac{\partial L}{\partial \dot{x}^i} (\eta^i - \chi \dot{x}^i) + \chi L - A. \quad (10)$$

This calculation will classify the spacetime given in Equation (2). The detail of which is given below.

3. SOLUTION-I

The general form of the time conformal Minkowski spacetime is:

$$ds^2 = e^{u(t)}(dt^2 - dx^2 - dy^2 - dz^2), \quad (11)$$

The system (9) has the following solution for any general function $u(t)$,

$$\begin{aligned} A &= c_1, & u &= u(t), & \chi &= c_2, & \eta_1 &= 0, \\ \eta_2 &= c_3y + c_4z + c_5 \\ \eta_3 &= -c_3x + c_6z + c_7, & \eta_4 &= -c_4x - c_6y + c_8 \end{aligned}$$

The action corresponding to the spacetime (11) admits the following Noether symmetries:

$$\begin{aligned} \mathbf{N}_1 &= \partial_s, & \mathbf{N}_2 &= \partial_x, & \mathbf{N}_3 &= \partial_y, & \mathbf{N}_4 &= \partial_z, \\ \mathbf{N}_5 &= y\partial_x - x\partial_y \\ \mathbf{N}_6 &= z\partial_x - x\partial_z, & \mathbf{N}_7 &= z\partial_y - y\partial_z \end{aligned} \quad (12)$$

\mathbf{N}_1 corresponds to the Lagrangian (4), \mathbf{N}_2 , \mathbf{N}_3 and \mathbf{N}_4 correspond to the linear momentum along x , y and z axes, respectively. Hence, the linear momentum along these axes conserved. \mathbf{N}_5 , \mathbf{N}_6 and \mathbf{N}_7 correspond to the conservation laws of angular momentum in xy , yz and zx planes respectively. The energy in the spacetime (11) is not conserved as we do not see the generator ∂_t in the set (12). The Einstein fields equations given in equation (1) for the spacetime given in Equation (11) takes the form:

$$\begin{aligned} -\frac{3}{4}u(t)_t^2 + e^{u(t)}\Lambda &= \kappa T_{00}, & u(t)_{tt} + \frac{1}{4}u_t^2 - \\ e^{u(t)}\Lambda &= \kappa T_{11} = \kappa T_{22} = \kappa T_{33} \end{aligned} \quad (13)$$

4. SOLUTION-II

The second solution of system (9) is:

$$ds^2 = (\alpha t + \beta)^\gamma(dt^2 - dx^2 - dy^2 - dz^2) \quad (14)$$

$$A = c_3, \quad \chi = c_1s + c_2, \quad \eta_1 = \frac{c_1(\alpha t + \beta)}{\alpha(\gamma + 2)},$$

$$\eta_2 = \frac{c_1x}{\gamma + 2} + c_4y + c_5z + c_6$$

$$\eta_3 = -c_4x + \frac{c_1y}{\gamma + 2} + c_7z + c_8$$

$$\eta_4 = -c_5x - c_7y + \frac{c_1z}{\gamma + 2} + c_9.$$

The corresponding Noether symmetries are:

$$\begin{aligned} \mathbf{N}_1 &= \partial_s, & \mathbf{N}_2 &= \partial_x, & \mathbf{N}_3 &= \partial_y, & \mathbf{N}_4 &= \partial_z, & \mathbf{N}_5 \\ & & & & & & & & = y\partial_x - x\partial_y \end{aligned}$$

$$\mathbf{N}_6 = z\partial_x - x\partial_z, \quad \mathbf{N}_7 = z\partial_y - y\partial_z$$

$$\mathbf{N}_8 = s\partial_s + \frac{\alpha t + \beta}{\alpha(\gamma + 2)}\partial_t + \frac{x}{(\gamma + 2)}\partial_x + \frac{y}{(\gamma + 2)}\partial_y + \frac{z}{(\gamma + 2)}\partial_z, \quad (15)$$

\mathbf{N}_1 , \mathbf{N}_2 , \mathbf{N}_3 , \mathbf{N}_4 , \mathbf{N}_5 , \mathbf{N}_6 and \mathbf{N}_7 do the same job as in Solution-I. \mathbf{N}_8 is the Noether symmetry generator corresponds to the scaling or the similarity transformation for the spacetime (14), it is the homothety of spacetime (14). The Einstein fields equations for the spacetime (14) take the form:

$$\begin{aligned} -\frac{3\gamma^2\alpha^2}{4(\alpha t + \beta)^2} + (\alpha t + \beta)^\gamma\Lambda &= \kappa T_{00}, \\ \frac{\gamma\alpha^2(\gamma - 4)}{4(\alpha t + \beta)^2} - (\alpha t + \beta)^\gamma\Lambda &= \kappa T_{11} = \kappa T_{22} = \kappa T_{33} \end{aligned} \quad (16)$$

Solving equation (16) for Λ , we have:

$$\Lambda = \frac{\kappa T_{00}}{(at+\beta)^\gamma} + \frac{3\alpha^2\gamma^2}{(at+\beta)^{2+\gamma}},$$

$$\Lambda = -\frac{\kappa T_{11}}{(at+\beta)^\gamma} + \frac{\alpha^2\gamma(\gamma-4)}{(at+\beta)^{2+\gamma}}.$$
(17)

Equation (17) shows that for positive γ the cosmological constant Λ is a decreasing function of time t .

5. SOLUTION-III

The third solution of system (9) is:

$$ds^2 = e^{\frac{t}{\alpha}}(dt^2 - dx^2 - dy^2 - dz^2).$$
(18)

$$A = 2c_1\alpha^2 e^{\frac{t}{\alpha}} + c_4, \chi$$

$$= \frac{1}{2}c_1s^2 + c_2s + c_3, \eta_1$$

$$= (c_1s + c_2)\alpha,$$

$$\eta_2 = c_5y + c_6z + c_7, \eta_3 = -c_5x + c_8z + c_9, \eta_4 = -c_6x - c_8y + c_{10}$$
(19)

The corresponding Noether symmetry generators are given as:

$$N_1 = \partial_s, N_2 = \partial_x, N_3 = \partial_y, N_4 = \partial_z,$$

$$N_5 = y\partial_x - x\partial_y$$

$$N_6 = z\partial_x - x\partial_z, N_7 = z\partial_y - y\partial_z,$$

$$N_8 = s\partial_s + \alpha\partial_t,$$

$$N_9 = \frac{s^2}{2}\partial_s + \alpha s\partial_t, A_9 = 2\alpha^2 e^{\frac{t}{\alpha}}.$$
(20)

N_8 contains the generator corresponding to the energy with some re-scaling term. The total conservation law (10) takes the form:

$$E_{rs} = E - sL,$$
(21)

where E_{rs} is the re-scale energy of the test particle in spacetime (18), while E is the exact energy of the test

particle and L is the Lagrangian density given in Equation (4). We see from Equation (21) that the energy of the particle decreasing linearly with respect to the proper time s , because of the curvature introduced in the spacetime (18) by the conformal factor $e^{\frac{t}{\alpha}}$. If we remove the effect of the conformal factor $e^{\frac{t}{\alpha}}$ from the spacetime (18), then the test particle will move forever with same energy given at the beginig. The curvature introduced by the time conformal factor will either accelerate the test particle or de-accelerate it. In our case it de-accelerate the test particle when it enters in the corresponding spacetime. Fig. 1 shows the decline of the energy of the test particle with passage of time. In the set of Noether symmetries (20), the symmetry generator N_9 comes with gauge function $A_9 = 2\alpha^2 e^{\frac{t}{\alpha}}$. This symmetry generator corresponds to the conformal transformation.

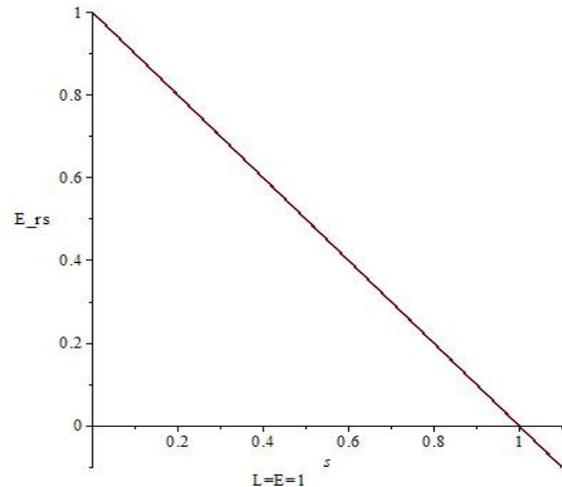


FIG. 1. ENERGY VERSUS TIME GRAPH

The Einstein field equations given in (1) for the metric (24) take the form:

$$-\frac{3}{4\alpha^2} + e^{\frac{t}{\alpha}\Lambda} = \kappa T_{00}, \quad \frac{1}{4\alpha^2} + e^{\frac{t}{\alpha}\Lambda} = \kappa T_{11} \\ = \kappa T_{22} = \kappa T_{33}. \quad (22)$$

From Equation (22) the cosmological constant Λ can be written as:

$$\Lambda = e^{-\frac{t}{\alpha}} \left(\kappa T_{00} + \frac{3}{4\alpha^2} \right), \\ \text{or } \Lambda = e^{-\frac{t}{\alpha}} \left(\kappa T_{11} - \frac{1}{4\alpha^2} \right). \quad (23)$$

Equation (23) shows that the cosmological constant decreases as time elapse, which agrees with the accelerated expansion of the universe.

6. SOLUTION-IV

The fourth solution which admits the maximum number of Noether symmetries in conformal Spacetime is:

$$ds^2 = (\alpha t + \beta)^{-2} (dt^2 - dx^2 - dy^2 - dz^2) \quad (24)$$

$$A = c_2, \chi = c_1,$$

$$\eta_1 = (c_3x + c_4y + c_5z + c_6)(\alpha t + \beta)$$

$$\eta_2 = \frac{\alpha}{2} (t^2 + x^2 - y^2 - z^2)c_3 + x(c_4y + c_5z + c_6)\alpha + c_3\beta t + c_8z + c_7y + c_9,$$

$$\eta_3 = \frac{\alpha}{2} (t^2 - x^2 + y^2 - z^2)c_4 + y(c_3x + c_5z + c_6)\alpha + c_4\beta t + c_{10}z + c_7y + c_{11},$$

$$\eta_4 = \frac{\alpha}{2} (t^2 - x^2 - y^2 + z^2)c_5 + z(c_3x + c_4y + c_6)\alpha + c_5\beta t - c_{10}y - c_8x + c_{12}, \quad (25)$$

$$\mathbf{N}_1 = \partial_s, \mathbf{N}_2 = \partial_x, \mathbf{N}_3 = \partial_y, \mathbf{N}_4 = \partial_z, \mathbf{N}_5 = y \partial_x - x \partial_y$$

$$\mathbf{N}_6 = z \partial_x - x \partial_z, \mathbf{N}_7 = z \partial_y - y \partial_z,$$

$$\mathbf{N}_8 = x(\alpha t + \beta) \partial_t + \left(\frac{\alpha}{2} (t^2 + x^2 - y^2 - z^2) + \beta t \right) \partial_x + xy\alpha \partial_y + xz\alpha \partial_z,$$

$$\mathbf{N}_9 = y(\alpha t + \beta) \partial_t + \left(\frac{\alpha}{2} (t^2 - x^2 + y^2 - z^2) + \beta t \right) \partial_y + xy\alpha \partial_x + yz\alpha \partial_z,$$

$$\mathbf{N}_{10} = z(\alpha t + \beta) \partial_t + \left(\frac{\alpha}{2} (t^2 - x^2 - y^2 + z^2) + \beta t \right) \partial_z + xz\alpha \partial_x + yz\alpha \partial_y,$$

$$\mathbf{N}_{11} = (\alpha t + \beta) \partial_t + x\alpha \partial_x + y\alpha \partial_y + z\alpha \partial_z, \quad (26)$$

The set (26) consists of eleven Noether symmetries and so eleven conservation laws hold in the spacetime (24). The first seven Noether symmetries do the same as did in the previous solutions, but $\mathbf{N}_8, \mathbf{N}_9, \mathbf{N}_{10}$ are the Noether symmetries which correspond to the conformal transformations. \mathbf{N}_{11} corresponds to the similarity transformation in the spacetime (24). The Einstein field equations given in Equation (1) for the metric given in Equation (24) take the form:

$$\frac{3\alpha^2}{(\alpha t + \beta)^2} - \frac{\Lambda}{(\alpha t + \beta)^2} = -\kappa T_{00} = \kappa T_{11} = \kappa T_{22} = \kappa T_{33}. \quad (27)$$

From equation (27) the cosmological constant can be written as:

$$\Lambda = (\alpha t + \beta)^2 \kappa T_{00} + 3\alpha^2. \quad (28)$$

7. CONCLUSION

The classification of the time conformal Minkowski spacetime is presented in this article. There are four classes of the time conformal Minkowski spacetime according to the Noether symmetries. These types of spacetimes admit, seven, eight, nine or eleven Noether symmetries. Therefore, there exist seven, eight, nine or eleven conservation laws for the Lagrangian of such spacetimes. In Equation (12), the Noether symmetry ∂_s corresponds to the Lagrangian L and the remaining symmetries are all isometries of the spacetime given in Equation (11). The Noether symmetry \mathbf{N}_9 given in Equation (15) corresponds to the scaling transformation in the corresponding spacetime (14) and is called homothety or homothetic vector field of the corresponding spacetime. The Noether symmetry \mathbf{N}_8 in Equation (20) is the Noether symmetry generator corresponds to the energy in spacetime (18) with the re-scaling term $s\partial_s$, while \mathbf{N}_9 is the symmetry generator corresponding to the conformal transformation. The time conformal spacetime given in Equation (24) is the one which admits maximum Noether symmetries and

the corresponding Lagrangian admits maximum conservation laws. The symmetry generators N_8 , N_9 and N_{10} correspond to the conformal transformations in the manifold of the spacetime (24). The Noether symmetry generator N_{11} given in Equation (26) is the symmetry generator corresponding to the similarity transformation (scaling).

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