

# Conformal and Disformal Structure of 3D Circularly Symmetric Static Metric in $f(R)$ Theory of Gravity

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## ABSTRACT

Conformal vector fields are treated as generalization of homothetic vector fields while disformal vector fields are defined through disformal transformations which are generalization of conformal transformations, therefore it is important to study conformal and disformal vector fields. In this paper, conformal and disformal structure of 3D (Three Dimensional) circularly symmetric static metric is discussed in the framework of  $f(R)$  theory of gravity. The purpose of this paper is twofold. Firstly, we have found some dust matter solutions of EFEs (Einstein Field Equations) by considering 3D circularly symmetric static metric in the  $f(R)$  theory of gravity. Secondly, we have found CKVFs (Conformal Killing Vector Fields) and DKVFs (Disformal Killing Vector Fields) of the obtained solutions by means of some algebraic and direct integration techniques. A metric version of  $f(R)$  theory of gravity is used to explore the solutions and dust matter as a source of energy momentum tensor. This study reveals that no proper DVFs exists. Here, DVFs for the solutions under consideration are either HVFs (Homothetic Vector Fields) or KVFs (Killing Vector Fields) in the  $f(R)$  theory of gravity. In this study, two cases have been discussed. In the first case, both CKVFs and DKVFs become HVFs with dimension three. In the second case, there exists two subcases. In the first subcase, DKVFs become HVFs with dimension seven. In the second subcase, CKVFs and DKVFs become KVFs having dimension four.

**Key Words:** Conformal Killing Vector Fields, Disformal Killing Vector Fields, Direct Integration Technique,  $f(R)$  Theory of Gravity.

## 1. INTRODUCTION

In physics, the relation between symmetries and conservation laws has fundamental importance. To classify the solutions of EFEs symmetries play an important role as with the help of symmetries one can explain geometry of spacetime [1]. Conformal symmetry which gives extra conservation laws than the other symmetry restrictions like Killing symmetry, homothetic

symmetry etc. plays a key role in Einstein theory of GR (General Relativity). It is to be noted that Killing symmetry and homothetic motions are particular forms of conformal symmetry. Conformal symmetry for different spacetimes has been discussed by different researchers [2-3]. Hall and Capocci [4] present a classification and conformal symmetry in 3D metric. 3D metrics play a key

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role in physics especially to black hole thermodynamics. Taking attention to black hole configuration, Berredo-Peixoto considered 3D gravity coupled to scalar field [5]. Banados et. al. [6] first discovered black hole solutions in 3D. After that a number of black hole solutions in 3D gravity have been found, a detail of which can be seen in [7-8] and the references therein. Bokhari et. al. [9] classified 3D circularly symmetric static metrics according to their isometries. Zhang et. al. [10] discussed black holes and gravitational waves in 3D  $f(R)$  gravity. Ali et. al. [11] investigated proper homothetic vector fields of 3D circularly symmetric static metric. Recently, Ramzan et. al. [12] discussed Ricci and Matter Collineations of Som-Roy Chaudhary Spacetime. Very recently, Larley [13] introduced the concept of DKVFs as a generalization of CKVFs. Actually, DKVFs are defined by means of disformal transformations which have many applications in TeVes theories for Mond [14], biometric theories of gravity [15], scalar tensor theories [16-19], disformal inflation [20], rainbow gravity [21] and many others. Keeping in mind the important applications of disformal transformations, we have used this idea in the  $f(R)$  theory of gravity which is introduced to justify current cosmic acceleration, problem of singularity, dark energy and dark matter issues etc. [22]. To tackle such type of issues, study of exact solutions in  $f(R)$  gravity has remained an important source of motivation for research. Therefore, working on 3D gravity theory plays a significant role in 3D physics. Our purpose in this paper is to find CKVFs and DKVFs by means of 3D geometry.

The rest of this paper is planned as follows: In section 2, we will give a brief derivation of dust matter solutions of EFEs by considering 3D circularly symmetric static metric in the  $f(R)$  theory of gravity as well as CKVFs and DKVFs of the resulting solutions. Section 3 is specified for results and discussion.

## 2. FIELD EQUATIONS AND THEIR SOLUTIONS IN $f(R)$ GRAVITY

Modified field equations in  $f(R)$  gravity are [23].

$$H_{ab} = kT_{ab} \tag{1}$$

Where  $H_{ab} = f(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \nabla_a \nabla_b f(R) + g_{ab} f''(R)$ ,  $f(R)$  is the function of Ricci scalar  $R$ ,  $f''(R) = \frac{d^2 f(R)}{dR^2}$ ,  $k$  is the coupling constant,  $T_{ab}$  is the standard energy momentum tensor and  $\nabla \equiv \nabla^a \nabla_a$  in which  $\nabla$  is the covariant derivative operator. To solve Equation (1), we consider 3D circularly symmetric static metric in the usual coordinates  $(t, r, \theta)$  given by  $(x^0, x^1, x^2)$  respectively [5].

$$ds^2 = e^{Y(r)} dt^2 + e^{Z(r)} dr^2 + r^2 d\theta^2 \tag{2}$$

where  $Y(r)$  and  $Z(r)$  are non-zero functions of the radial coordinate  $r$ . Non-zero components of Christoffel symbols for the above metric Equation (2) are:

$$\Gamma_{01}^0 = \frac{1}{2} Y', \Gamma_{00}^1 = \frac{1}{2} Y' e^{Y-Z}, \Gamma_{11}^1 = \frac{1}{2} Z', \Gamma_{22}^1 = r e^{-Z}, \text{ and } \Gamma_{12}^2 = \frac{1}{r}$$

Similarly, non-zero components of Ricci tensors for the above metric Equation (2) are:

$$\left. \begin{aligned} R_{00} &= \frac{1}{4r} [2rY'' + rY'^2 - rY'Z' + 2Y'] e^{Y-Z} \\ R_{11} &= \frac{1}{4r} [2rY'' + rY'^2 - rY'Z' + 2Z'] \\ R_{22} &= \frac{r}{2} [Y' - Z'] e^{-Z} \end{aligned} \right\} \tag{3}$$

where prime denotes the derivative with respect to the radial coordinate  $r$ . To explore solutions of Equation (1), we use dust matter as a source of energy momentum tensor i.e.  $T_{ab} = \rho S_a S_b$  [23], where  $\rho$  is the matter density and  $S_a$  is the velocity vector defined as  $S_a = - e^{Y(r)/2} \delta_a^0$ . Ricci scalar for the above metric Equation (2) is given by:

$$R = \frac{1}{2r} [2rY'' + rY'^2 - rY'Z' + 2Y' - 2Z'] e^{-Z} \quad (4)$$

Using Equations (2-3) in Equation (1) after some calculations, we obtain following two Equations (5-6):

$$\left[ \frac{YF}{2r} - \frac{YF'}{2} + \frac{ZF}{2r} + F'' - \frac{ZF'}{2} \right] e^{-Z} + k\rho = 0 \quad (5)$$

$$\left[ \frac{Y'F}{2} + \frac{Y'^2 F'}{4} - \frac{YZF'}{4} - \frac{YF'}{2} + \frac{ZF}{2r} + \frac{F'}{r} \right] e^{-Z} + k\rho = 0 \quad (6)$$

Our purpose is to solve Equations (5-6) which are highly non-linear, therefore it is very difficult to find the solution. Moreover, Equations (5-6) has three unknowns namely Y, Z and F. Therefore, we need some extra condition to find the values of these unknowns. For this, we assume that  $Y' = 0$  so that Equations (5-6) after subtraction reduce into following Equation (7):

$$\left[ F'' - \frac{ZF'}{2} - \frac{F'}{r} \right] = 0 \quad (7)$$

To solve Equation (7) we use power law model of  $f(R)$  i.e.  $F(R) \propto f_0 R^m$  [24], where  $f_0$  and  $m$  are arbitrary constants. Hence,

$$\left[ R'' + (m-1) \frac{R'^2}{R} - \frac{ZR'}{2} - \frac{R'}{r} \right] = 0 \quad (8)$$

with Ricci scalar  $R = \frac{-Z'}{r} e^{-Z}$ . Assuming  $e^{Z(r)} = r^\alpha$  where  $\alpha$  is any real number, we have from Equation (8) that  $\alpha(\alpha-2)r^{\alpha-4}[3\alpha-8+2(\alpha-2)(m-1)]=0$  and the Ricci scalar  $R$  takes the form  $R = \alpha r^{\alpha-2}$ . Here, following cases arise:

**Case (a):**  $[3\alpha - 8 + 2(\alpha-2)(m-1)] = 0$  but  $\alpha(\alpha-2)r^{\alpha-4} \neq 0$ .

**Case (b):**  $\alpha(\alpha-2)r^{\alpha-4} = 0$  but  $[2\alpha - 8 + (\alpha-2)(m-1)] \neq 0$ .

We will discuss each case in detail.

**Case (a):** In this case, we have  $[3\alpha - 8 + 2(\alpha-2)(m-1)] = 0$  but  $\alpha(\alpha-2)r^{\alpha-4} \neq 0$ , number of solutions could be obtained for different values of  $\alpha$ . For example, a solution is obtained for  $\alpha=4$  which implies that  $m=0$ . In this case metric Equation (2) takes the form

$$dS^2 = \alpha_1 dt^2 + r^4 dr^2 + r^2 d\theta^2 \quad (9)$$

and the value of Ricci scalar is  $R=4r^2$ . Now, we are interested to find CKVFs of the above metric Equation (9). A vector field  $X$  is said to be a conformal Killing vector field if [25].

$$L_X g_{ab} = 2\eta g_{ab} \quad (10)$$

Where  $\eta = \eta(r)$ ,  $X$ ,  $g_{ab}$  and  $L$  represents conformal factor, vector field, metric tensor and Lie derivative operator respectively. Using Equation (10) in Equation (9), one obtains the following six first order non-linear PDEs (Partial Differential Equations):

$$X^0_{,0} = \eta \quad (11)$$

$$-\alpha_1 X^0_{,1} + r^{-4} X^1_{,0} = 0 \quad (12)$$

$$r^2 X^2_{,0} - \alpha_1 X^0_{,2} = 0 \quad (13)$$

$$-2X^1 + r X^1_{,1} = \eta r \quad (14)$$

$$X^1_{,2} + r^6 X^2_{,1} = 0 \quad (15)$$

$$X^1 + r X^2_{,2} = \eta r \quad (16)$$

Solving Equations (12-13) and Equation (15) by direct integration technique, we get the following information:

$$\left. \begin{aligned} X^0 &= \frac{r}{\alpha_1} \int A^1(t, \theta) d\theta + \frac{1}{\alpha_1} \int r^{-4} A^3_t(t, r) dr + A^2(t, \theta) \\ X^1 &= r^4 \iint A^1(t, \theta) dt d\theta - r^6 \int A^2_r(r, \theta) d\theta + A^3(t, r) \\ X^2 &= \frac{1}{r} \int A^1(t, \theta) dt + A^2(r, \theta) \end{aligned} \right\} \quad (17)$$

Where  $A^1(t,\theta)$ ,  $A^2(r,\theta)$  and  $A^3((t,r))$  are arbitrary functions of integration to be determined. To find the values of these unknown functions of integration, we use the remaining Equations (11,14) and Equation (16). Skipping lengthy but straightforward calculations, we obtain  $\eta=c_1$  which means that conformal vector fields become homothetic vector fields which are:

$$\left. \begin{aligned} X^0 &= c_1 t + c_3 \\ X^1 &= -c_1 r \\ X^2 &= 2c_1 \theta + c_8 \end{aligned} \right\} \quad (18)$$

where  $c_1, c_3, c_8 \in \mathfrak{R} \setminus \{0\}$ . Now, we find DKVFs of the metric Equation (9) which is defined as [13]

$$L_K g_{ab} = 2m g_{ab} + 2p V_a V_b \quad (19)$$

where ‘ $m$ ’ and ‘ $p$ ’ are scalar fields and  $V$  is called null-like vector. Equivalently,

$$L_K g_{ab} = 2m g_{ab} + 2p g_{ac} g_{bd} V^c V^d \quad (20)$$

Assuming a null vector  $v = \left( \sqrt{\frac{2}{\alpha_1}}, r^2, r^{-1} \right)$  for the metric Equation (9) and using Equation (20) one can obtain a system of six first order non-linear PDEs:

$$K_{,0}^0 = m - 2p \quad (21)$$

$$\alpha_1 r^4 K_{,1}^0 - K_{,0}^1 = 2pr^2 \sqrt{2\alpha_1} \quad (22)$$

$$\alpha_1 K_{,2}^0 - r^2 K_{,0}^2 = 2pr \sqrt{2\alpha_1} \quad (23)$$

$$-2K^1 + rK_{,1}^1 = 2r(m + p) \quad (24)$$

$$K_{,2}^1 + r^6 K_{,1}^2 = 2pr^3 \quad (25)$$

$$K^1 + rK_{,2}^2 = r(m + p) \quad (26)$$

Where  $m$  and  $p$  are functions of the radial coordinates ‘ $r$ ’. Equations (21,24 and 26) gives the following information:

$$\left. \begin{aligned} K^0 &= t(m - 2p) + B^3(r, \theta), K^1 = r^2 \int \frac{m+p}{r^2} dr + r^2 B^1(t, \theta) \\ K^2 &= \theta(m - 2p) - r\theta \int \frac{m+p}{r^2} dr + \int B^1(t, \theta) d\theta + B^2(t, r) \end{aligned} \right\} \quad (27)$$

where  $B^1(t,\theta)$ ,  $B^2(t,r)$  and  $B^3(r,\theta)$  are functions of integration to be determined. Again, using some algebraic and direct integration techniques and avoiding tedious calculations, one finds that  $p=0$  and  $m=c_1$  which implies that the DKVFs reduce into homothetic vector fields and are given in Equation (18).

**Case (b):** In this case, we have  $\alpha(\alpha-2)r^{\alpha-4} = 0$  but  $[2\alpha - 8 + 2(\alpha - 2)(m-1)] \neq 0$ . Now, if  $\alpha(\alpha-2)r^{\alpha-4} = 0$ , then two subcases arise: **(bi)**  $\alpha=0$  and **(bii)**  $\alpha=2$  For the subcase **(bi)** i.e.  $\alpha=0$  we have the following metric:

$$ds^2 = \alpha_1 dt^2 + dr^2 + r^2 d\theta^2 \quad (28)$$

Using the procedure as given in case (a), one finds that  $\eta=c_2$  which means that CKVFs become HVFs which are:

$$\left. \begin{aligned} X^0 &= c_2 t + \frac{r}{\alpha_1} (c_7 \cos \theta + c_8 \sin \theta) + c_4 \\ X^1 &= c_2 r + t(c_7 \cos \theta + c_8 \sin \theta) + c_{10} \cos \theta + c_{11} \sin \theta \\ X^2 &= \frac{1}{r} [t(-c_7 \sin \theta + c_8 \cos \theta) + (c_{10} \sin \theta + c_{11} \cos \theta)] + c_9 \end{aligned} \right\} \quad (29)$$

where  $c_2, c_4, c_7, c_8, c_9, c_{10}, c_{11} \in \mathfrak{R} \setminus \{0\}$ . Next, we find DKVFs of the above metric Equation (28). For doing this, we define a null vector  $v = \left( \sqrt{\frac{2}{\alpha_1}}, r^2, r^{-1} \right)$  for the metric Equation (28) and using the definition of DKVF given in Equation (20), one finds that  $p=0$  and  $m=c_2$  which means that the DKVFs reduce into homothetic vector fields and are given in Equation (29). For the subcase **(bii)**  $\alpha=2$  we have the following metric:

$$dS^2 = -\alpha_1 dt^2 + r^2 dr^2 + r^2 d\theta^2 \quad (30)$$

Again using the definitions given in Equations (10 and 20) and skipping the details one finds that  $p = m = \eta = 0$  which means that both CKVFs and DKVFs become KVFs which are:

$$\left. \begin{aligned} X^0 = K^0 = c_4, X^1 = K^1 = 2r\theta c_1 - c_2 r \\ X^2 = K^2 = -\left(\frac{r^2\theta^2}{r^2}\right)c_1 + c_2\theta + c_3 \end{aligned} \right\} \quad (31)$$

where  $c_1, c_2, c_3, c_4 \in \mathfrak{R}/\{0\}$ .

## 5. CONCLUSION

In this paper, CKVFs and DKVFs structure of 3D circularly symmetric static metric in the  $f(R)$  theory of gravity is presented. The purpose of this paper was twofold. Firstly, we have found some dust matter solutions of EFEs in the  $f(R)$  theory of gravity and then we have extended our analysis by finding CKVFs and DKVFs of the resulting solutions. This study consists of two cases. Studying each case in detail, we find that in the case (a), both the CKVFs and DKVFs reduce into HVFs having dimension three see Equation(18). From these three HVFs two are KVFs and one is proper HVF. In the case (bi), both the CKVFs and DKVFs become HVFs having dimension seven; see Equation (29). Out of these seven HVFs six are KVFs one is proper HVF. In the case (bii), both the DKVFs and CKVFs become KVFs having dimension four; see Equation (31). Moreover, the use of null-like disformal transformation shows that the geometry of an object is changed and the background and foreground light cones coincides in the null direction. Actually, these are the preliminary results on DKVFs which may further be extended in four dimensions.

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