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A Family of Binary Approximating Subdivision Schemes based on Binomial Distribution

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ABSTRACT

A simplest way is introduced to generate a generalized algorithm of univariate and bivariate subdivision schemes. This generalized algorithm is based on the symbol of uniform B-splines subdivision schemes and probability generating function of Binomial distribution. We present a family of binary approximating subdivision schemes which has maximum continuity and less support size. Our proposed family members P_4, P_5, P_6, and P_7, have C^7, C^9, C^{11} and C^{13} continuities respectively. In fact, we use Binomial probability distribution to increase the continuity of uniform B-splines subdivision schemes up to more than double. We present the complete analysis of one family member of proposed schemes and give a visual performance to check smoothness graphically. In our analysis, we present continuity, Holder regularity, degree of generation, degree of reproduction and limit stencils analysis of proposed family of subdivision schemes. We also present a survey of high continuity subdivision schemes. Comparison shows that our proposed family of subdivision schemes gives high continuity of subdivision schemes comparative to existing subdivision schemes. We have found that as complexity increases the continuity also increases. In the last, we give the general formula for tensor product surface subdivision schemes and also present the visual performance of proposed tensor product surface subdivision schemes.

Key Words: Binary,Approximating Subdivision Schemes,Uniform B-Splines,Binomial Distribution, Tensor Product.

1. INTRODUCTION

CAGD (Computer Aided Geometric Design) is a field which is related to Applied Mathematics. In this arena, we follow procedures for the shapes that are usually used in the design, computer graphics for their mathematical study. The important tool of CAGD is subdivision schemes. Subdivision schemes are the iterative formulas which are used for generation of smooth curves and surfaces form initial polygon or initial mesh.

Developing new subdivision schemes for curve and surface design has its own importance. In the field of subdivision schemes de Rham [1] and Chaikin [2] are regarded as the pioneers. They developed the corner cutting schemes which generate C^1 limit curve. Now a day wide variety of approximating and interpolating schemes have been proposed in the literature which possess shape parameters. Dyn et. al. [3] presented a

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four-point subdivision scheme which has C^2 continuity. Siddiqi and Ahmad [4-6] offered three, five and six points approximating subdivision schemes having C^2, C^4 and C^6 limit curves respectively when the shape parameter w = 1/4.

Mustafa et. al. [7] presented 6-point approximating subdivision schemes which give C^6 continuity when the shape parameter $w \in (-1.6, 1.3)$. Mustafa et. al. [8] presented a generalized proof of the smoothness of 6-point interpolatory scheme. They proved that 6-point interpolatory scheme introduced by [8] have C^3 continuity when the shape parameter $w \in [0.0139, 0.0143]$. Ghaffar et. al. [9] presented 3-point approximating subdivision scheme which is C^3 continuous at $\mu = 1$. Ghaffar and Mustafa [10] presented a family of even point ternary approximating subdivision schemes. One family member of 6-point ternary approximating subdivision scheme gives C^7 continuity at $w = 1/12$. Tan et. al. [11] presented a new four-point shape-preserving subdivision scheme with two shape parameters which has C^3 continuity at $\alpha = 1$, $\beta = 32$.

Hameed and Mustafa [12] presented a family of univariate and bivariate binary approximating subdivision schemes using Lane- Riesenfeld algorithm.

Zheng et. al. [13] presented designing multi-parameter curve subdivision schemes with high continuity. They offered a technique to increase the continuity of subdivision scheme. The way of increasing the continuity is multiplying the symbol with $(1+z/2)^k$ factor, after multiplication they get C^{n+k}continuous subdivision schemes. But the demerits of this technique are: when they multiply symbol with factor then complexity, support size and mathematical computation of subdivision schemes are also increased. After this literature, we offer a technique that gives high continuity subdivision schemes, less complexity and support size.

In this paper, we construct a parametric family of subdivision schemes which have high continuity, less complexity and support size. The construction of proposed family based on probability density function of binomial distribution and symbol of uniform B-splines subdivision schemes. We give the comparison of continuity analysis with uniform B-splines and other existing binary subdivision schemes which have high continuity. This paper is also a survey of high continuity subdivision schemes.

The paper is organized as follows. In Section 2, we present construction of a family of binary approximating subdivision schemes. Analysis of the proposed family is presented in Section 3. Section 4 is for general formulae of a family of tensor product surface subdivision schemes. Applications and conclusions are drawn in Sections 5 and 6 respectively.

2. CONSTRUCTION OF A FAMILY OF SCHEMES

In this section, we will present an algorithm for parametric families of subdivision schemes. There are two subsections, one is for a family of Binomial distribution based subdivision schemes and other is for a family of binary approximating subdivision schemes which is based on the symbol of uniform B-splines and probability generating function of binomial distribution.

2.1 Binomial Distribution Based Subdivision Schemes

Here we will construct binary approximating subdivision schemes using the binomial distribution. The Binomial distribution is given by:

$$a_i = \binom{n}{i} p^i (1-p)^{n-i}, i = 0,1,...,n \tag{1}$$

where the variable i with $0 \leq i \leq n$ and the parameter n are any positive integers. The parameter p is a probability of successive events; the range of p is $0 \leq p \leq 1$. The distribution describes the probability of exactly i successes in n (fixed number) trials if the probability of success in a single trial is p (we sometimes also use 1-p, the probability for a failure). Bernoulli first presented in a work which was published in [14]. After substituting the different values of $n, n \geq 1$ in Equation (1), we get $(n + 1)$-point binary approximating subdivision schemes. The general form of $(n+1)$-point binary approximating subdivision schemes is:

$$\begin{aligned} f_{2i}^{k+1} &= a_0 f_i^k + a_1 f_{i+1}^k + ... + + a_{n-1} f_{i+n-1}^k + a_n f_{i+n}^k \\ f_{2i+1}^{k+1} &= a_n f_i^k + a_{n-1} f_{i+1}^k + ... + + a_1 f_{i+n-1}^k + a_o f_{i+n}^k \end{aligned} \tag{2}$$

For $n = 1$ in Equation (1), we get $a_0 = (1-p)$ and $a_1 = p$ then Equation (2) takes the form:

$$\begin{aligned} f_{2i}^{k+1} &= (1-p) f_i^k + p f_{i+1}^k \\ f_{2i+1}^{k+1} &= p f_i^k + (1-p) f_{i+1}^k \end{aligned} \tag{3}$$

If we put $p = 1/2$ in Equation (3), then we get a well-known Chaikin [2] corner cutting scheme. Similarly, for n = 2, we get $a_0 = (1-p)^2$, $a_1 = 2p(1-p)$ and $a_2 = p$ then Equation (2) becomes:

$$\begin{aligned} f_{2i}^{k+1} &= (1-p)^2 f_i^k + 2p(1-p) f_{i+1}^k + p^2 f_{i+2}^k \\ f_{2i+1}^{k+1} &= p^2 f_i^k + 2p(1-p) f_{i+1}^k + (1-p)^2 f_{i+2}^k \end{aligned}$$

We can generate a family of $(n+1)$-point binary approximating subdivision schemes by using the different values of n and $p \in (0,1)$.

2.2 Family of Binary Approximating Subdivision Schemes with High Continuity

This subsection is for the general symbol of a family of binary approximating subdivision schemes with one shape parameter. Probability density function of binomial distribution and $(n + 1)$ the degree polynomial uniform B-splines for all $n \geq 1$ are used to construct the general symbol of the family of binary approximating subdivision schemes. The symbol of uniform B-splines [15] subdivision schemes are given as:

$$A_n(z) = \frac{(1+z)^{n+1}}{2^n} \tag{4}$$

The probability generating function of binomial distribution [16] is given as:

$$B_n(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i \tag{5}$$

The symbol defined in Equation (4) has C^{n-1} continuous subdivision schemes. In order to increase the continuity of uniform B-splines and generate a new family of approximating subdivision schemes, we take a product of symbol of uniform B-splines schemes (4) and probability density function of the binomial distribution (5). The general symbol of a family of approximating subdivision schemes is defined as:

$$P_n(z) = A_n(z) B_n(z)$$

The simplest form of the symbol is:

$$P_n(z) = \frac{(1-p)^{n-1}}{2^n} \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i \tag{6}$$

This is the general formula for a family of binary approximating subdivision schemes with one parameter.

Our proposed family of binary approximating subdivision schemes with one parameter has high continuity, less support size and many existing subdivision schemes are particular cases of our proposed subdivision schemes. We get the family members of approximating subdivision schemes corresponding to each value of n in Equation (6), and for different values of p can generate the different subdivision schemes. Table 1 gives the complexity and mask of proposed family of approximating subdivision schemes.

3. ANALYSIS OF THE SCHEMES

This section aims to present the analysis of a proposed family of binary approximating subdivision schemes. Here we only present the analysis of one family member P_2 of binary subdivision schemes, the analysis of other family members is similar.

3.1 Convergence and Smoothness Analysis

Continuity is an important property of subdivision schemes. Continuity of a subdivision scheme refers to the differentiability of the limit curve/surface produced by subdivision process. A high continuous subdivision scheme gives a more smooth limit curve. We use Laurent polynomial (symbol) method [17] to calculate integer class continuity of the P_n-schemes. In general $P_n(z)$ in Equation (6) is C continuous but when we substitute the value of P=1/2, $P_n(z)$ have maximum continuity. Table 2 gives the comparison of continuity analysis of uniform B-splines schemes $A_n(z)$, proposed $P_n(z)$ schemes in general and proposed $P_n(z)$ schemes at p = 1/2.

Theorem-1: The 3-point binary scheme corresponding to the symbol P_2 is C^2 continuous.

Proof: The Laurent polynomial of the 3-point binary scheme is obtained by substituting n=2 in Equation (6), which is:

$$P_2(z) = \frac{1}{4}(1+z)^3\left((1-p)^2 + 2p(1-p)z + p^2z^2\right) \quad (7)$$

TABLE 1. HERE WE PRESENT THE MASK OF FAMILY OF BINARY APPROXIMATING SUBDIVISION SCHEMES CORRESPONDING TO DIFFERENT VALUES OF *n*, HERE *m* SHOWS COMPLEXITY OF THE SCHEMES (i.e. 2-3,….-POINT SCHEMES)

n	*m*	Mask
1	2	$P_1 = \frac{1}{2}\left[p, p+1, 2-p, 1-p\right]$
2	3	$P_2 = \frac{1}{4}\left[p^2, p^2+2p, 1-2p^2+4p, 3-2p^2, 3-4p+p^2, 1-2p+p^2\right]$
3	4	$P_3 = \frac{1}{8}\left[p^3, p^3+3p^2+6p^2+3p-3p^3, 1+9p-3p^3, 3p^3+6p+4-12p^2,\right.$ $\left.6-6p+3p^3-3p^2, -p^3+6p^2-9p+4, 1-3p-p^3+3p^2\right]$
4	5	$P_4 = \frac{1}{16}\left[p^4, p^4+4p^3, -4p^4+8p^3+6p^2, 4p-4p^4-8p^3+18p^2, 16p+6p^2-24p^3+6p^2-24p^3+6p^4+1, 6p^4+5-30p^2+20p,\right.$ $\left.10-30p^2-4p^4+24p^3, 6p^2-20p+10-4p^3, 18p^2-8p^3+5-16p+p^4, 1-4p-4p^3+p^4+6p^2\right]$

This implies

$$P_2(z)=\left(\frac{1+z}{2}\right)^2 b(z) \tag{8}$$

where b(z) = (1+z) c(z), c(z) = ((1-p)^2 + 2p(1-p) z + p^{2}z^2).

$$\|c(z)\|_\infty = \max\left\{\left|(1-p)^2\right| + \left|p^2\right|, \left|2p(1-p)\right|\right\}$$

The condition for $\|c(z)\|_\infty < 1$ is $0 < p < 1$. Therefore by [13], if $\|c(z)\|_\infty < 1$, then c(z) is contractive and b(z) is converges. If b(z) is convergent then scheme corresponding to P$_2$(z) is C^{2}continuous.

HolderRegularityAnalysis: Holder regularity is an extension of convergence and continuity which gives more information about any scheme. Lower and upper bounds on Holder continuity are calculated by using Rioul's method [18]. Moreover, exact upper bounds on Holder continuity can also be derived by using Floater and Muntingh algorithm [19]. In Table 3, we summarized the results of lower and upper bounds of Holder continuity which depends on the value of *p*. Moreover, we obtain exact Holder regularity when p = 1/2.

Theorem-2: Holder regularity of 3-point binary scheme corresponding to the symbol P$_2$(z) is 3 at p = 1/2.

TABLE 2. HERE WE SHOW THE COMPARISON OF CONTINUITY ANALYSIS OF UNIFORM B-SPLINES SCHEMES CORRESPONDING TO A$_n$(z), PROPOSED SCHEMES CORRESPONDING TO P$_n$(z) IN GENERAL AND PROPOSED SCHEMES CORRESPONDING TO P$_n$(z) at p = 1/2 FOR DIFFERENT n

n	A_n(z	P_n(z)	P_n(z) at w = 1/2
1	C_0	C_1	C_2
2	C_1	C_2	C_3
3	C_2	C_3	C_5
4	C_3	C_4	C_7
5	C_4	C_5	C_9
6	C_5	C_6	C_{11}
7	C_6	C_7	C_{13}

TABLE 3. HERE WE PRESENT THE HOLDER REGULARITY ANALYSIS OF SOME MEMBERS OF THE FAMILY OF BINARY P$_n$ SCHEMES FOR n = 1,2, AND 3. HERE n, k, L$_h$, U$_h$ AND E$_h$ ARE ANY POSITIVE INTEGER, COMMON FACTORS, LOWER BOUND OF HOLDER REGULARITY, THE UPPER BOUND OF HOLDER REGULARITY AND EXACT HOLDER REGULARITY AT p=1/2

n	*k*	L_h	U_h	E_h
1	2	$\left\{\begin{matrix} 2(1-p)^2 & if & 0<p<0.5 \\ 2p & if & 0.5<p<1 \end{matrix}\right\}$	$\left\{\begin{matrix} 2(1-p) & if & 0<p<0.5 \\ 2p & if & 0.5<p<1 \end{matrix}\right\}$	2
2	3	$\left\{\begin{matrix} 2(1-p)^2 & if & 0\le p\le 0.3 \\ 2p(1-p) & if & 0.3\le p\le 0.6 \\ 2p^2 & if & 0.6\le p\le 0.9 \end{matrix}\right\}$	$\left\{2p^2+2(p-1)^2\right\}$	3
3	4	$\left\{\begin{matrix} \beta_1 & if & 0<p<0.3 \\ \beta_2 & if & 0.3<p\le 0.7 \\ \beta_3 & if & 0.7<p<1 \end{matrix}\right\}$	$\left\{\begin{matrix} \beta_4 & if & 0<p\le 0.5 \\ \beta_5 & if & 0.5<p<1 \end{matrix}\right\}$	6

Thus the limit stencil of scheme corresponding to $P_2(z)$ is given by $P_2 = \{l, l_1, l_2, l_3\}$. Which complete the proof.

3.4 Survey and Comparison with Existing Schemes

Here we will present a survey of high continuity parametric subdivision schemes. We also show that our proposed family of binary approximating subdivision schemes gives the highest continuity as compare to existing parametric subdivision schemes. Table 5 present a survey and comparison with proposed family of approximating subdivision schemes.

4. TENSOR PRODUCT FAMILY OF APPROXIMATING SUBDIVISION SCHEMES

In this section, we will present the generalized formulae for a family of approximating tensor product surface subdivision schemes. Uniform B-splines subdivision schemes for tensor product surface is defined as:

$$A_n(z_1, z_2) = \frac{(1+z_1)^{n+1}(1+z_2)^{n+1}}{2^{2n}} \quad (11)$$

Probability density function of Binomial distribution for tensor product surfaces subdivision schemes is defined as:

$$B_n(z_1, z_2) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z_1^i \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z_2^i \quad (12)$$

The general symbol of a family of approximating subdivision schemes is defined as:

$$P_n(z_1,z_2) = A_n(z_1,z_2)\, B_n(z_1,z_2) \quad (13)$$

which is the general formula for tensor product of a proposed family of approximating subdivision schemes, where $A_n(z_1,z_2)$ is defined in Equation (11) and $B_n(z_1,z_2)$ is defined in Equation (12) with $p \in (0,1)$. By substituting the different values of n in Equation (13), we can obtain the family members of tensor product surface subdivision schemes.

TABLE 5. SHOWS THE COMPARISON OF CONTINUITY ANALYSIS OF EXISTING PARAMETRIC SUBDIVISION SCHEMES AND PROPOSED FAMILY OF SCHEMES E,C AND P_n DENOTE THE EXISTING SCHEMES, CONTINUITY OF EXISTING SCHEMES AND CONTINUITY OF PROPOSED SCHEMES At p=1/2 RESPECTIVELY

E	C	P^n
3-Point [4]	C^2 at w = 1	C^3
3-Point [9]	C^3 at μ = 1	C^3
4-Point [3]	C^2	C^5
4-Point [21]	C^4	C^5
4-Point [22]	C^5 at w = 1/4	C^5
5-Point [5]	C^4 at w = 1/4	C^7
5-Point [23]	C^5 at w = 1/128	C^7
5-Point [11]	C^3 at α = 1, β = 32	C^7
6-Point [4]	C^4 at w = 1/4	C^9
6-Point [6]	C^9 with 6 paprameter	C^9
6-Point [24]	C^7 at w$\in$ = 1/12	C^9
6-Point [10]	C_8 if w$\in$(-1.6,1.3)	C^9
6-Point [7]	C^3 if w [0.0139, 0.0143]	C^9
6-Point [8]	C^2	C^9
6-Point [25]	C^2	C^9
6-Point [26]	C^9	C^{11}

Proof: From Equation (7) we have $b_0 = (1-p)^2$, $b_1 = 4p(1-p)$, $b_0 = 2p^2$, k = 3, m = 2 and b = 2 thus q = 0,1,2,B_o,B_1 and B_2 are the matrices with elements

$(B_0)_{i,j} = b_{2+i-2j}$, $(B_1)_{i,j} = b_{3+i-2j}$, and $(B_2)_{i,j} = b_{4+i-2j}$

Where i,j = 1 and 2. Thus B_0, B_1 and B_2 are given by:

$$B_0\begin{pmatrix} b_1 & 0 \\ b_2 & b_0 \end{pmatrix}, B_1\begin{pmatrix} b_2 & b_0 \\ 0 & b_1 \end{pmatrix}, \text{and } B_2\begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}$$

By [14], the Holder regularity is given by $r = k - \log_b(\mu)$, where μ is the joint spectral radius of the matrices B_0, B_1 and B_2 that is $\mu = \rho(B_0, B_1, B_2)$. For bounds on Holder regularity, we calculate

$$\max\{\rho(B_0),\rho(B_1),\rho(B_2)\} \le \mu \le \max\{\|B_0\|_\infty,\|B_1\|_\infty,\|B_2\|_\infty\}$$

Since μ is bounded from below by the spectral radii and from above by the norm of the metrics B_0, B_1, and B_2. We have

$$\max\{2(1-p)^2, 4p(1-p), 2p^2\} \le \mu \le \max\{4p(1-p), 2p^2+2(p-1)^2, 2p^2\}$$

The value of μ depends on the range of *w*.

$$\begin{Bmatrix} 2(1-p)^2 & if & 0 \le p \le 0.3 \\ 4p(1-p) & if & 0.3 \le p \le 0.6 \\ 2p^2 & if & 0.6 \le p \le 0.9 \end{Bmatrix} \le \mu \le \{2p^2 + 2(p-1)^2\} \quad (9)$$

By using Equation (9), we can calculate the lower and upper bounds of Holder regularity for different values of p. The exact Holder regularity of 3-point binary scheme corresponding to the symbol $P_2(z)$ is $r = 3 - \log_2(1) = 3$ at p = 1/2, which completes the proof.

where

$$\beta_1 = 2-6p+6p^2-2p^3, \beta_2 = 2p(1-p)\left(-\frac{3}{2}+\frac{1}{2}\sqrt{9+32p^2-32p}\right)$$

$$\beta_3 = 2p(1-p)\left(-\frac{3}{2}+\frac{1}{2}\sqrt{9+32p^2-32p}\right), \beta_4 = 6\left|p^2(1-p)\right|+2\left|(p-1)^3\right|$$

and

$$\beta_5 = 6\left|p(1-p)^2\right|+2\left|p^3\right|$$

3.2 Generation and Reproduction Analysis

The subdivision scheme with symbol $P_n(z)$ reproduces polynomials of degree *d* with respect to the parameterizations with $\tau = p_n^{'}(1)/2$ if and only if

$$p_n^k(-1) = 0 \text{ and } p_n^k(1) = 2\prod_{j=0}^{k-1}(\tau - j), \ k = 0,1,2,...,d \quad (10)$$

Polynomial reproduction of degree *d* requires the polynomial generation of degree *d* [13]. Support of the schemes is calculated by [20]. Table 4 summarizes the results of support, generation degree, reproduction degree, shift parameter and parametrization of proposed schemes corresponding to $P_n(z)$ and generation degree at p = 1/2 of proposed schemes corresponding to $P_n(z)$.

Theorem-3: Generation degree of 3-point binary scheme corresponding to the symbol $P_2(z)$ is 2.

TABLE 4. HERE WE PRESENT AN ANALYSIS OF SOME MEMBERS OF THE FAMILY OF BINARY P_n SCHEMES FOR n = 1,2,3 AND 4. HERE n, S, G_d, R_d, τ AND PARE ANY POSITIVE INTEGER, SUPPORT, GENERATION DEGREE, REPRODUCTION DEGREE, SHIFT PARAMETER, AND PARAMETRIZATION

n	S	G_d	R_d	τ	P	G_d at p = 1/2
1	3	1	1	τ = 1	Primal/Dual	2
2	5	2	1	τ = (4p + 3)/2		4
3	7	3	1	τ = 2+ 3p		6
4	9	4	1	τ = (8p + 5)/2		9

Proof: The Laurent polynomial of 3-point binary scheme defined in Equation (7) can be written as:

$$P_2(z) = \left(\frac{1+z}{2}\right)^{2+1} b(z)$$

where $b(z) = 2((1-p)^2 + 2p(1-p)z + p^2z^2)$.

Hence generation degree is 2.

Theorem-4: The 3-point binary scheme corresponding to the symbol $P_2(z)$ reproduces polynomial of degree 1 concerning the dual parameterizations and also gives the primal parametrization at p = 1/4

Proof: By taking the first derivative of Equation (7) and put z = 1, we get $(P_2)'(1) = 4p + 3$ This implies that $\tau = 4p+3/2$, so the scheme corresponding to the symbol $P_2(z)$ has dual parametrization. This scheme also gives the primal parametrization at p = 1/4. We can easily verify that the first and second derivatives of $P_2(z)$ if z = - 1 are equal to 0. Further, we can also verify (10) for k = 0 and 1. This completes the proof.

3.3 Limit Stencil Analysis

The limit stencil is a way to obtain a point on the limit curve in the form of the original control points.

The procedure for calculating the limit stencils is presented in [17]. Here we present the analysis of limit stencils of one family member.

Theorem-4: Limit stencil of 3-point binary scheme corresponding to the symbol $P_2(z)$ is $P_2 = \{l, l_1, l_2, l_3\}$, where

$$l = \frac{(5p+1)p^3}{6(p^2-p+2)}$$

$$l_1 = \frac{(5p^3+11p^2-18p-4)p}{6(p^2-p+2)}$$

$$l_2 = \frac{4p^4-31p^3+45p^2-13p-6}{6(p^2-p+2)}$$

and

$$l_3 = \frac{(4p^2-11p+6)(p^2-2p+1)}{6(p^2-p+2)}$$

Proof: The subdivision matrix of the scheme corresponding to the symbol $P_2(z)$ defined by Equation (7) is:

$$A = \begin{pmatrix} \frac{p^2}{4}+\frac{p}{2} & -\frac{p^2}{2}+\frac{3}{4} & \frac{p^2}{4}-\frac{p}{2}+\frac{1}{4} & 0 \\ \frac{p^2}{4} & -\frac{p^2}{2}+p+\frac{1}{4} & -\frac{p^2}{4}-p+\frac{3}{4} & 0 \\ 0 & \frac{p^2}{4}+\frac{p}{2} & -\frac{p^2}{2}+\frac{3}{4} & \frac{p^2}{4}-\frac{p}{2}+\frac{1}{4} \\ 0 & \frac{p^2}{4} & -\frac{p^2}{2}+p+\frac{1}{4} & \frac{p^2}{4}-p+\frac{3}{4} \end{pmatrix}$$

Eigenvalues of A are $\lambda_0 = 1$, $\lambda_1 = 1/2$, $\lambda_2 = 1/4$ and $\lambda_3 = -p^2/2 + p/2$. The matrix of eigenvectors corresponding to eigenvalues is:

$$B = \begin{pmatrix} 1 & \frac{4p-5}{1+4p} & \frac{7p^2-16p+9}{7(2+7p)} & \frac{(p^2-2p+1)(p^4-5p^3+6p-4)}{p^3(-3p-p^2-1+p^3)} \\ 1 & -\frac{4p-3}{1+4p} & \frac{7p^2-10p+3}{p(2+7p)} & -\frac{(p^2-2p+1)(p^2-5p+4)}{p(-3p-p^2-1+p^3)} \\ 1 & \frac{4p-1}{1+4p} & \frac{7p-4}{p(2+7p)} & -\frac{(p^2-2p+1)(3+p)}{-3p-p^2-1+p^3} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We can define the diagonal matrix D as:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -\frac{p^2}{2}+\frac{p}{2} \end{pmatrix}$$

By diagonalization of the matrix A, we get A + BDB^{-1}, where

$$B=\begin{pmatrix} \frac{p^3(5p+1)}{6(p^2-p+2)} & -\frac{p(5p^3+11p^2-18p-4)}{6(p^2-p+2)} & c & g \\ \frac{p^3(4p+1)}{2(p^2-p+2)} & -\frac{p(p^2+2p-2)(1+4p)}{2(p^2-p+2)} & d & h \\ \frac{p^3(7p+2)}{3(2p^2-2p+1)} & -\frac{p(p^2+2p-1)(2+7p)}{3(2p^2-2p+1)} & e & i \\ a & b & f & j \end{pmatrix}$$

Where

$$a=\frac{p^3(-3p-p^2-1+p^3)}{2(2p^6-6p^5-16p^3+14p^2+13p^4-7p+2)}$$

$$c=-\frac{5p^4-31p^3+45p^2-13p-6}{2(p^2-p+2)}$$

$$b=-\frac{3p^3(-3p-p^2-1+p^3)}{2(2p^6-6p^5-16p^3+14p^2+13p^4-7p+2)}$$

$$g=-\frac{(5p^2-11p+6)(p^2-2p+1)}{6(p^2-p+2)}$$

$$d=-\frac{(p^2-2p+1)(p-1)(1+4p)}{2(p^2-p+2)}$$

$$h=-\frac{(p^2-2p+1)(p-1)(1+4p)}{2(p^2-p+2)}$$

$$f=-\frac{3p^3(-3p-p^2+1+p^3)}{2(2p^6-6p^5-16p^3=14p^2-7p+2)}$$

$$e=-\frac{p(2+7p)(p^2-4p+2)}{3(2p^2-2p+1)}$$

$$j=-\frac{p^3(-3p-p^2-1+p^3)}{2(2p^6-6p^5-16p^3+14p^2+13p^4-7p+2)}$$

$$i=-\frac{p(2+7p)(p^2-2p+1)}{3(2p^2-2p+1)}$$

Since A = BDB^{-1} this implies A^j = BD^{j}B^{-1},

Where

$$D^j=\begin{pmatrix} 1^j & 0 & 0 & 0 \\ 0 & 0.5^j & 0 & 0 \\ 0 & 0 & 0.25^j & 0 \\ 0 & 0 & 0 & \left(-\frac{p^2}{2}+\frac{p}{2}\right)^j \end{pmatrix}$$

Therefore

$$D^\infty=\lim_{j\to\infty}D^j=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

By [17], we have

$$\begin{pmatrix} f_{-2}^\infty \\ f_{-1}^\infty \\ f_0^\infty \\ f_1^\infty \end{pmatrix}=\begin{pmatrix} l & l_1 & l_2 & l_3 \\ l & l_1 & l_2 & l_3 \\ l & l_1 & l_2 & l_3 \\ l & l_1 & l_2 & l_3 \end{pmatrix}\begin{pmatrix} f_{-2}^\infty \\ f_{-1}^0 \\ f_0^0 \\ f_1^0 \end{pmatrix}$$

Where

$$l=\frac{(5p+1)p^3}{6(p^2-p+2)}$$

$$l_1=-\frac{(5p^3+11p^2-18p-4)p}{6(p^2-p+2)}$$

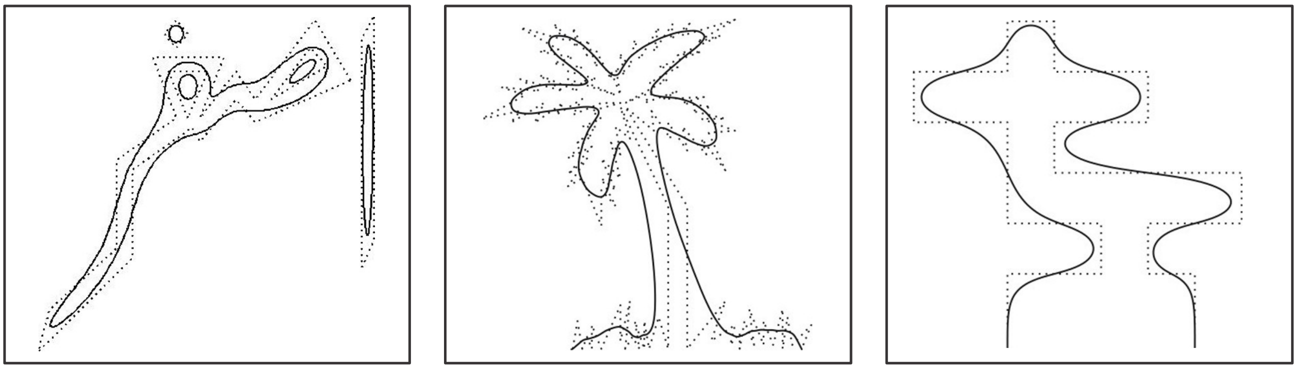
$$l_2=-\frac{5p^4-31p^3+45p^2-13p-6}{6(p^2-p+2)}$$

and

$$l_3=-\frac{(5p^2-11p+6)(p^2-2p+1)}{6(p^2-p+2)}$$

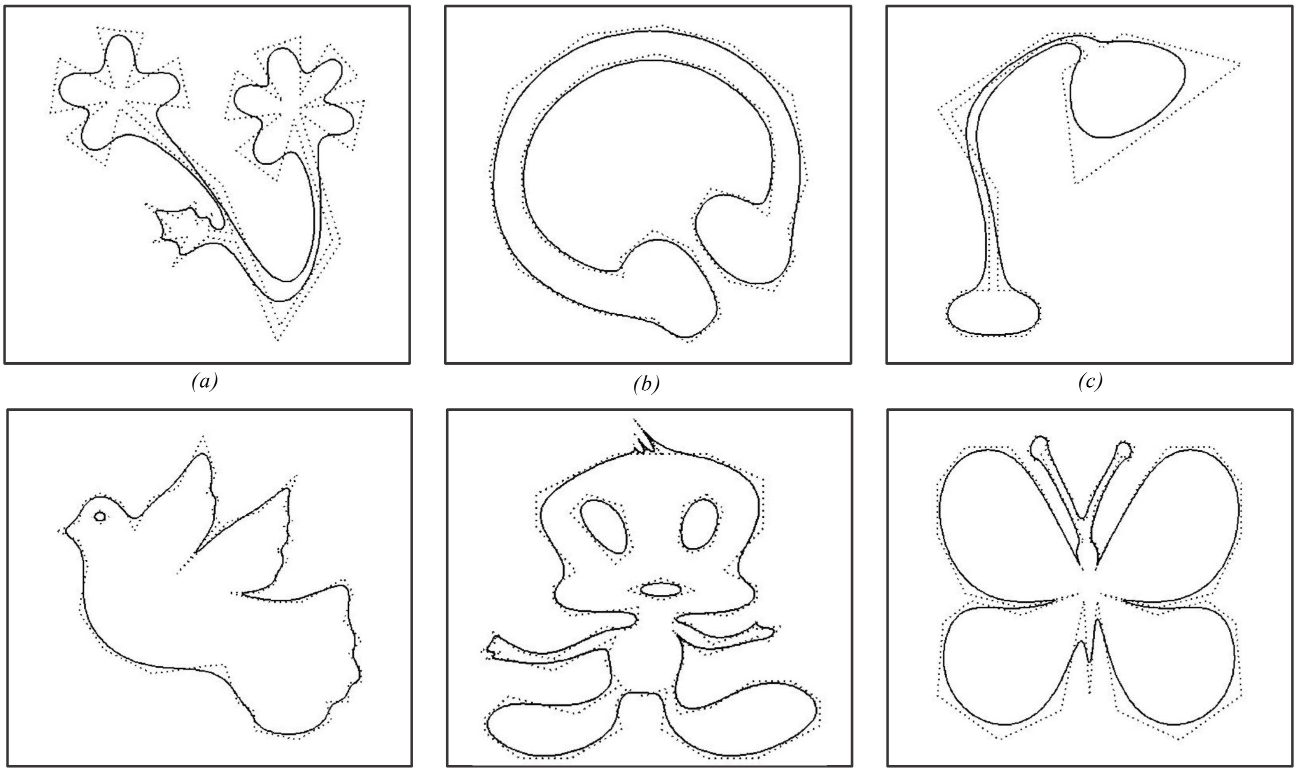
5. APPLICATIONS

In this section, we will show the visual performance of proposed schemes. These examples show smooth curves which pass through a set of given points. The control polygons are drawn by a dashed line and the smooth curves obtained by our proposed schemes by a full line. Fig. 1(a-c) is an example of limit curves for open and closed polygons, Fig. 2(a-f) present limit curves for close polygons and Fig. 3(a-c) give the comparison of limit curves generated by the schemes corresponding to P_2, P_3 and P_4. After comparison, we see that P_4 gives more smooth curves as compare to P_2+ and P_3 at p = 1/2 Figs. 4-5(a-e) are the visual performances of surface subdivision schemes P_3.



(a) *(b)* *(c)*

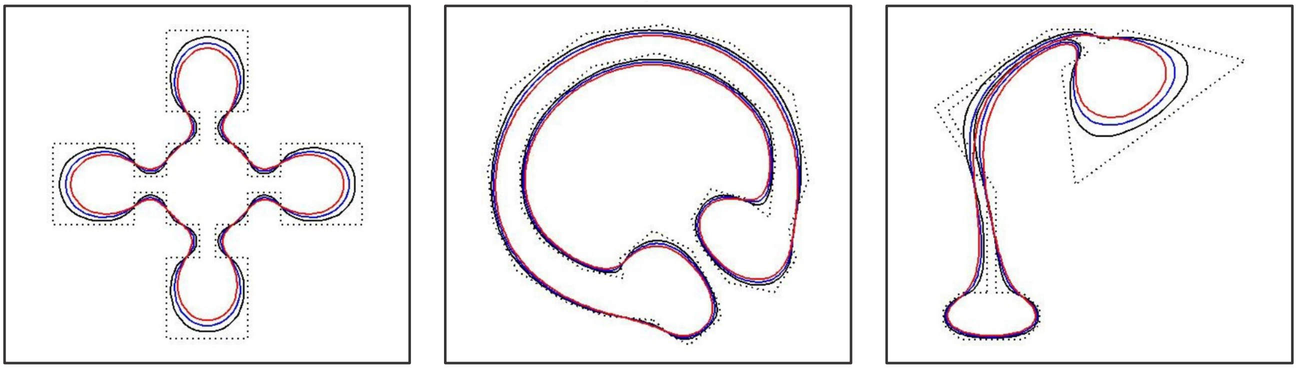
FIG. 1. PRESENT LIMIT CURVES FOR OPEN AND CLOSED POLYGONS PRODUCED BY A SCHEME CORRESPONDING TO P_2



(a) *(b)* *(c)*

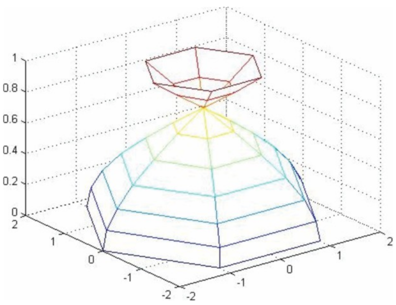
(d) *(e)* *(f)*

FIG. 2. PRESENT LIMIT CURVES FOR CLOSE POLYGONS PRODUCED BY A SCHEME CORRESPONDING TO P_2

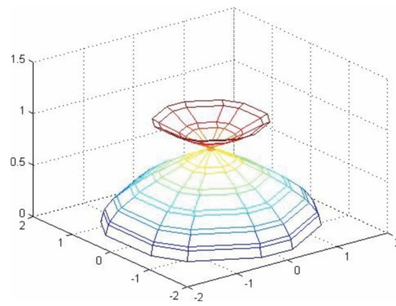


(a) *(b)* *(c)*

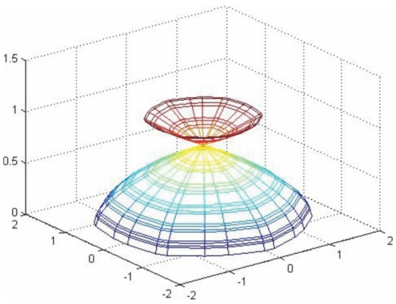
FIG. 3. PRESENT COMPARISON OF LIMIT CURVES FOR CLOSE POLYGONS PRODUCED BY A SCHEME CORRESPONDING TO P_2, P_3 AND P_4. HERE BLACK DOTTED LINES SHOWS THE INITIAL POLYGON, BLACK SOLID LINE IS THE LIMIT CURVE GENERATED BY A SCHEME CORRESPONDING TO P_2, THE BLUE SOLID LINE IS THE LIMIT CURVE GENERATED BY A SCHEME CORRESPONDING TO P_3 AND RED SOLID LINE IS THE LIMIT CURVE GENERATED BY A SCHEME CORRESPONDING TO P_4



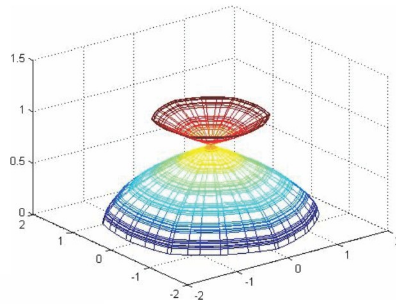
(a) INITIAL SURFACE



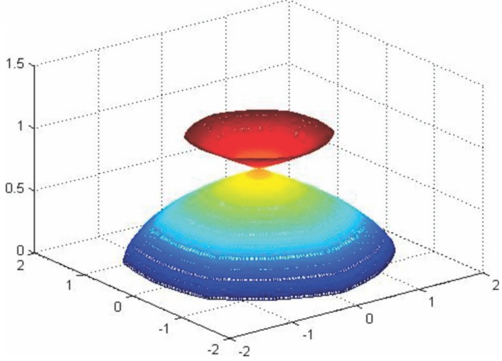
(b) LEVEL 1



(c) LEVEL 2

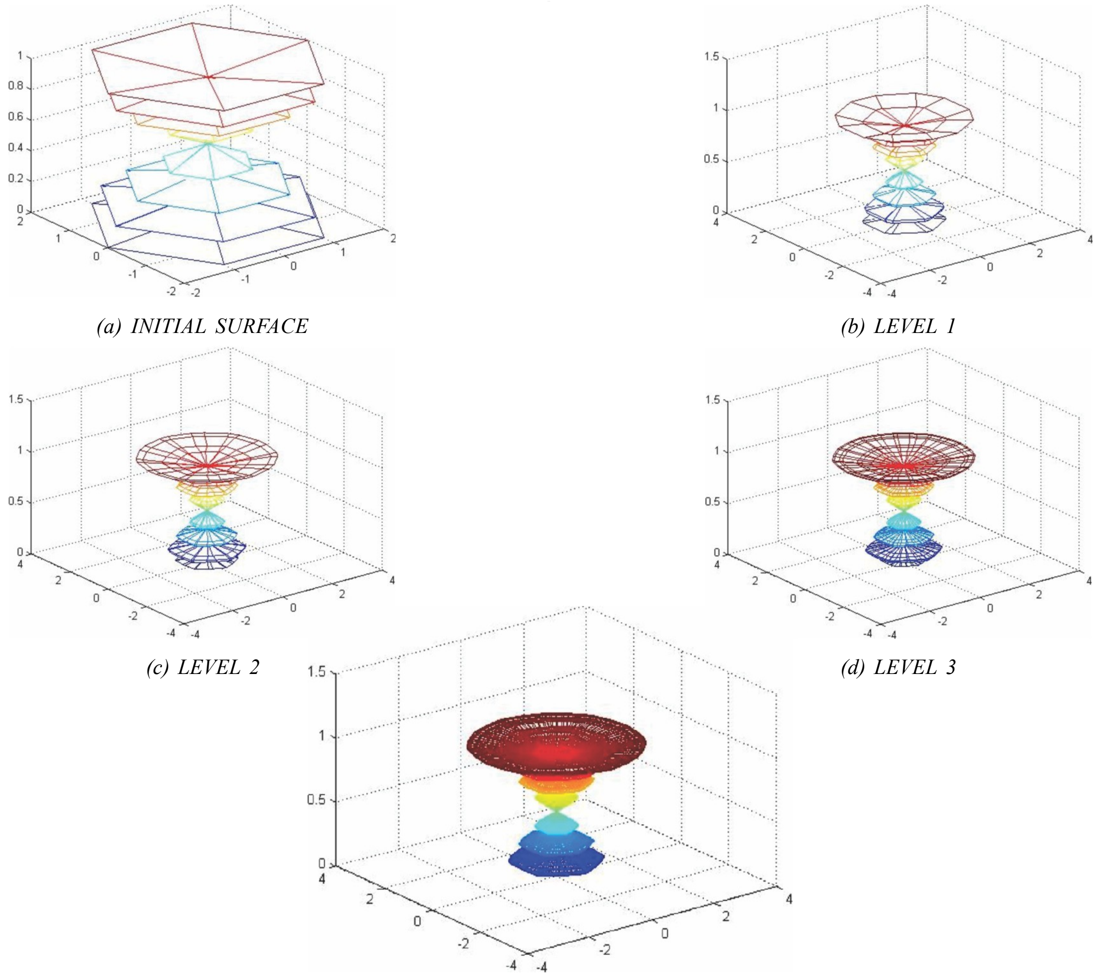


(d) LEVEL 3



(e) LIMIT SURFACE

FIG. 4. PRESENT LIMIT CURVES FOR CLOSE POLYGONS PRODUCED BY A SCHEME CORRESPONDING TO P_3 SURFACES SUBDIVISION SCHEME



(a) INITIAL SURFACE

(b) LEVEL 1

(c) LEVEL 2

(d) LEVEL 3

(e) LIMIT SURFACE

FIG. 5. PRESENT LIMIT CURVES FOR CLOSE POLYGONS PRODUCED BY A SCHEME CORRESPONDING TO P_3 SURFACES SUBDIVISION SCHEME

6. CONCLUSION

In this paper, we have presented a generalized algorithm to introduce a parametric family of binary approximating subdivision schemes with high continuity. In our proposed algorithm, the continuity of uniform B-splines subdivision schemes increased up to more than double. We also give the complete analysis of one family member of proposed schemes. Visual performance shows the smoothness of a family of schemes graphically. Tensor product surface formula of proposed family of schemes is also presented. This paper also contains a survey of high continuity parametric subdivision schemes.

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