Performance Evaluation of Controlled Arrival Rate System through Matrix Geometric Method Using Transient Analysis

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ABSTRACT

Performance evaluation based on transient analysis has significant importance to those systems which never reached at equilibrium. In this paper, we develop and analyze the system through controlled arrival rate using internal limits. The system is modeled with Markovian and phase-type distributions for various internal limits and queue size. The transient analysis is carried out for the systems in vector domain through matrix geometric method and Runge-kutta procedure. The comparisons of results of transient analysis for Markovian and phase-type controlled arrival rate systems are also done.

Key Words: **Performance, Evaluation, Equilibrium, Controlled Arrival Rate, Runge-Kutta Procedure.**

1. INTRODUCTION

Performance evaluation of queueing systems as a QBD (Quasi Birth and Death) process, having structured Markov chain can efficiently be done through matrix geometric method [1]. For infinite and finite capacity queue based on Markovian distribution and phase type distributions can be analyzed for various performance measures using matrix geometric method [2].

Markovian and phase-type queueing systems can efficiently be solved through matrix geometric method as compared to other numerical solution techniques and main block matrices can easily be obtained without constructing the Markov chain [3]. The QBD process has a special tridiagonal structure in its transition matrix and it can be solved efficiently through the matrix geometric method, which is one of the matrix analytic methods [4,5].

Transient analysis is very essential in those queueing systems in which queues never reached at equilibrium [6]. There are several methods available to analyze the queueing system transient behavior but they are complex in nature. The Bessel functions are commonly used to derive the expressions of the system which are complicated to solve [7]. The transient analysis of the queueing systems can efficiently be done through matrix geometric method and Runge-kutta procedure [8].

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Bodrog, et. al. [9] initial transient period of system is analyzed and the initial transient comprehensive framework is discussed. The transient probabilities obtained from inverting functions discussed in [10].

In this paper, we analyze the controlled arrival rate for transient behavior system using Markovian and Phase type distributions in vector domain through matrix geometric method.

The structure of this paper is as follows: Matrix geometric method, Markovian, phase type distributions and their comparison are briefly presented in Section 2. In Section 3, controlled arrival rate system using Markovian , phase type distributions and their solutions along with case study are discussed. Results and conclusion are discussed in Sections 4 and 5.

2. METHODS AND TERMINOLOGIES

In this section, first we discuss the matrix geometric method and then Markovian and phase type distributions along with their comparison.

2.1 Matrix Geometric Method

The vector form technique which provides simplest and efficient procedure to obtain the Markov chain state probability in vectors is called matrix geometric method. The existence of geometric relation in state probability vector of stochastic process leads the technique applicable. It is defined as:

$$\Pi_1 = \Pi_0 R$$

the rate matrix R is the essential matrix which defines the system.

2.2 Markovian Distribution

If the probability distribution of the process depends only on the present state is called a Markovian distribution and it follows the Markov process which can be represented by the Markov chain. The Markov chain using Markovian distribution behaves like a quasi birth and death process and defines as:

$$p_{ij(t)} = p_r \left[X(t_n) = X(t_{n-1} + t) \right]$$

2.3 Phase Type Distribution

If the Poisson processes occurs in phases which are from one or more systems inter related are called phase type distribution. Each phase in a sequence is a stochastic process. The following are the special cases of a continuous phase-type distributions.

2.3.1 Hypoexponential Distribution

The hypoexponential distribution is the generalization of the Erlang distribution by having the different rates for each transition. The phase type distribution of hypoexponential distribution is given by:

$$\alpha = (1,0,0,...,0) T = \begin{bmatrix} -r_1 & r_1 & 0 & 0 & \dots & 0 \\ 0 & -r_2 & r_2 & 0 & \dots & 0 \\ 0 & 0 & -r_3 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -r_k \end{bmatrix}$$

$$Q = \left[\begin{array}{cccccc|c} -r_1 & r_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -r_2 & r_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -r_3 & r_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -r_k & r_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

2.3.2 Hyperexponential Distribution

Hyperexponential distribution has two parameters, different probabilities α_i and different rates r_k. The hyperexponential phase type distribution is given by:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k) T = \begin{bmatrix} -r_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -r_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & -r_3 & 0_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -r_k \end{bmatrix}$$

$$Q = \left[\begin{array}{cccccc|c} -r_1 & 0 & 0 & 0 & \cdots & 0 & r_1 \\ 0 & -r_2 & 0 & 0 & \cdots & 0 & r_2 \\ 0 & 0 & -r_3 & 0 & \cdots & 0 & r_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -r_k & r_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

The comparison between Markovain and phase type distribution is shown in Table 1.

3. CONTROLLED ARRIVAL RATE SYSTEM

The controlled arrival rate system is shown in Fig. 1. The arrival rate of the system depends on the customer present in the system. If the customers present in the system within the predefined limit, the rate of arrival of customer is normal, otherwise rate is reduced.

There are two predefined limits in the system, s_1 and s_2. The customers are allowed with arrival rate λ_1 up to limit s_2 once the limit reached the arrival rate will reduce to λ_2 until the customers in the system reached at limit s_1. The system is designed and develops with following distributions.

3.1 Controlled Arrival Rate Model with Markovian Distribution

The arrival and service processes of the controlled arrival rate system are modeled as a Markovian distribution. The structured Markov of the system is in the form of quasi birth and death process as shown in Fig. 2.

3.2 Controlled Arrival Rate Model with Phase Type Distribution

The service process of the controlled arrival rate system is modeled by two different phase type distributions, hypo and hyperexponential distributions, where as the arrival process is modeled as a Markovian distribution. These systems are shown in Fig. 3(a-b).

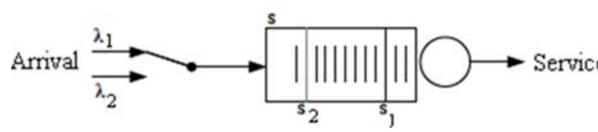

FIG. 1. CONTROLLED ARRIVAL RATE SYSTEM

TABLE 1. COMPARISON BETWEEN MARKOVAIN AND PHASE TYPE DISTRIBUTION

Distribution	Mean	Variance	PDF	CDF	Cv	Parameter
Markovian	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\lambda e^{-\lambda t}$	$1-e^{-\lambda t}$	Equal to 1	λ
Hypoexponential	$\sum_{i=1}^{n} \frac{1}{\lambda_t}$	$\sum_{i=1}^{n} \frac{1}{\lambda_t^2}$	$-\alpha e^{x\theta}\theta 1$	$1-\alpha e^{x\theta 1}$	Less than 1	$\lambda_1, \lambda_2, ... \lambda_n$
Hyperexponential	$\sum_{i=1}^{n} \frac{p_i}{\lambda_t}$	$\sum_{i=1}^{n} \frac{p_i}{\lambda_t^2}(2-p_i)$	$\sum_{i=1}^{n} p_i \lambda_i e^{-\lambda t}$	$1-\sum_{i=1}^{n} p_i \lambda_i e^{-\lambda t}$	Greater than 1	$\lambda_1, p_1, \lambda_2, p_2, ... \lambda_n p_n$

3.3 Application: Case Study

In communication network, buffer or queue is a special feature of the system where incoming or outgoing data/ signals are temporary stored and waiting prior to service. In real systems, theses storage areas are limited in capacity which results overloading and congestion problems. Controlling of data or signal arrival rate signal is an efficient technique through which we can overcome the problem of overloading and congestion. These network buffers handles two type of data arrivals distributions, one type of data distribution in which arrival of data directly enters the system buffer where as in other type of distribution, arrival of data in the system can only enter the system after passes from different stages. These two data arrival distributions can efficiently be modeled through Markovian and phase type distributions. The transient analysis of this network buffer provides a significant knowledge of overall system behavior when system is not reached in steady state. Above two approaches defined in Sections 3.1 and 3.2 can efficiently be applied to telecommunication network buffer to analyze the transient behavior of the buffer in which arrival rate of data/signal is controlled to overcome the overloading and congestion.

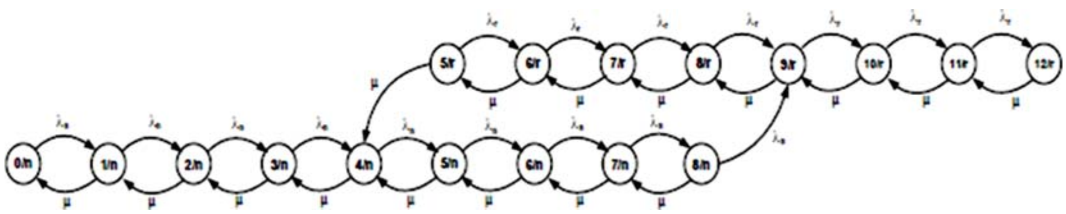

FIG. 2. STRUCTURED MARKOV CHAIN WITH MARKOVIAN DISTRIBUTION

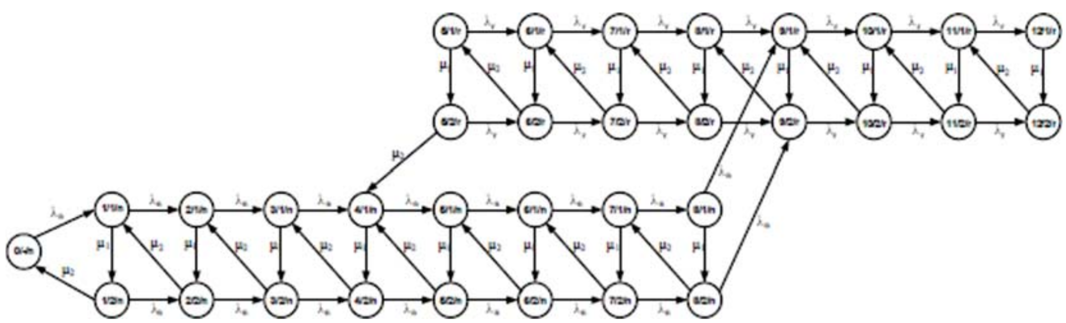

FIG. 3(a). STRUCTURED MARKOV CHAIN WITH HYPOEXPONENTIAL DISTRIBUTION

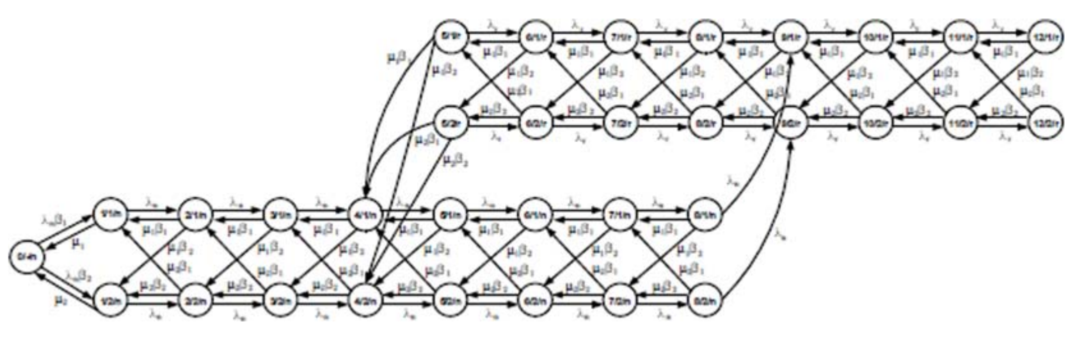

FIG. 3(b). STRUCTURED MARKOV CHAIN WITH HYPEREXPONENTIAL DISTRIBUTION

4. TRANSIENT ANALYSIS

The analytical program for transient analysis is developed in VC++ using matrix geometric method and Runge-kutta procedure in vector form for controlled arrival rate system.

The comparison of the transient analysis of a controlled arrival rate system with Markovian and phase type distributions is shown in Fig. 4. The graph shows the comparison with smaller and larger difference between the s_1 and s_2 limits. Here, it is observed that very small oscillation occurs and system becomes stabilized soon for all three models for small dfference between the s_1 and s_2. For larger difference between the s_1 and s_2, more oscillations are observed for these systems and a significant difference between the oscillations of these systems is also observed. It is easily analyze thate system with hypoexponential distribution has more oscillations as compared to the other two systems and the system with Markovian distribution is more oscillated as compared to the hyperexponential system. The hyperexponential system has smaller oscillation and stabilizes soon as compared to the other two systems.

The obtained results are verified and validated through the results obtained by solving the systems using system linear equations and Chapman Kolmogrov equations as shown in Fig. 5.

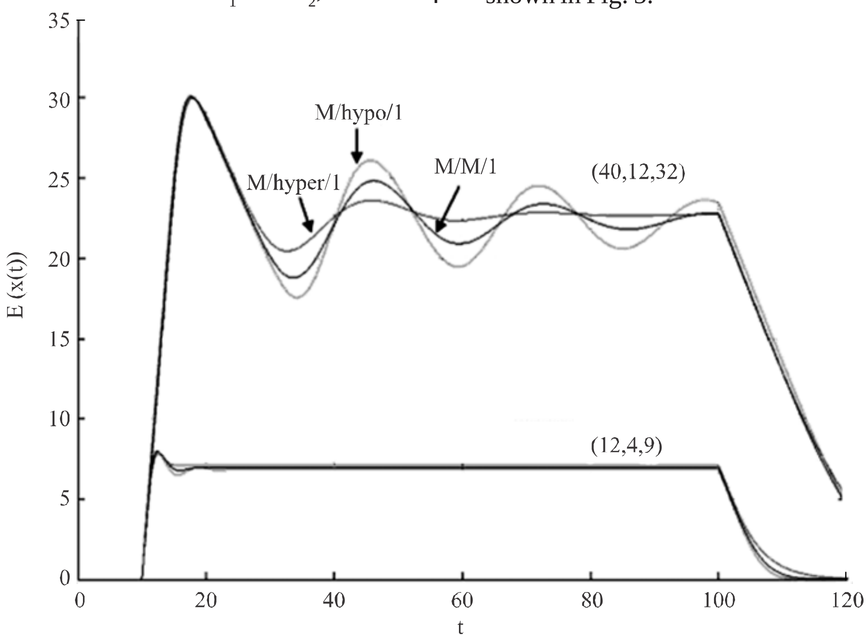

FIG. 4. TRANSIENT ANALYSIS: MEAN NUMBER IN THE SYSTEM: COMPARISON OF TRANSIENT ANALYSIS WITH MARKOVIAN, HYPO AND HYPEREXPONENTIAL DISTRIBUTION

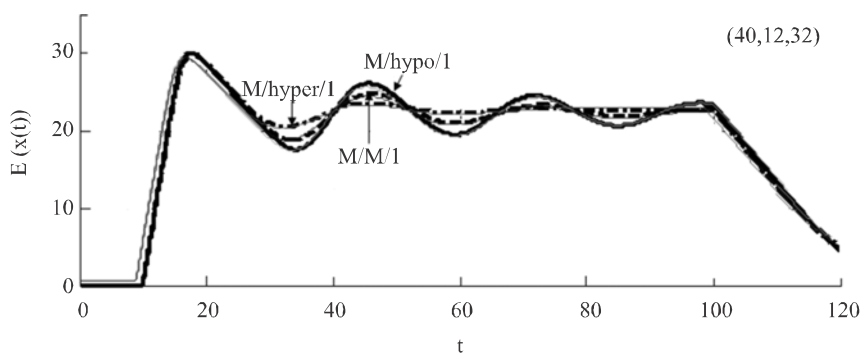

FIG. 5. TRANSIENT ANALYSIS: MEAN NUMBER IN THE SYSTEM: RESULTS VALIDATION

5. CONCLUSIONS

In this paper, we used a Markovian and phase type distributions to model the controlled arrival rate system. The system is solved through Markovian and two phase type distributions called hypo and hyperexponential distributions.

The vector form structured Markov chains are constructed and analyzed using matrix geometric method along with Runge-kutta procedure for the transient behavior. The comparison of the obtained results for system using Markovian and phase type distributions shows that hypoexponential system has more oscillations as compared to the Markovian and hyperexponential systems. The obtained results are also validated through basic classical methods.

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