Flow Time Analysis of Load Management Late Arrival Discrete Time Queueing System with Dual Service Rate Using Hypogeometrical Distribution

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ABSTRACT

Flow time analysis is a powerful concept to analyze the flow time of any arriving customer in any system at any instant. A load management mechanism can be employed very effectively in any queueing system by utilizing a system which provides probability of dual service rate. In this paper, we develop and demonstrate the flow and service processes transition diagram to determine the flow time of a customer in a load management late arrival state dependent finite discrete time queueing system with dual service rate where customers are hypogeometrically distributed. We compute the probability mass function of each starting state and total probability mass function. The obtained analytical results are validated with simulation results for varying values of arrival and service probabilities.

Keywords: **Flow Time Process, Late Arrival , Service Process, Probability Mass Function, State Dependent, Load Management.**

1. INTRODUCTION

The flow time process is very effective to evaluate the flow time of any customer in the system. The Flow time of a customer is determined by analyzing the test customer. The flow time process provides the information about the time and path of the customer which spent and follows in the system. Newly entered customer flow time is influenced by already present customers in the system. It is very helpful and advantageous to develop a flow time process transition diagram to see the whole process of each customer in the system [1,2].

Modeling a queueing system is an important tool in studying and performance analysis of complex systems [3]. A load management mechanism can be employed very effectively in any queueing system by utilizing a system which provides probability of dual service rate. Although, queueing theory is rich in context of its availability in the literature but many times we deal non-Markovian systems using some approximations to help and understand it properly [4]. Phase type distribution is a major tool in creation and analysis of such queuing systems. A load management mechanism to avoid the congestion in any communication system especially in digital system can efficiently be modeled through discrete time queueing system where arrival of customers can be modeled as discrete time phase distributed.

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Discrete-time queueing is based on the assumption that the time is divided into fixed length of intervals (equidistant) called slots [1,6]. Discrete-time system allows multiple events during one time unit called a slot. The common causes of the change that takes place in any queueing system are the events when a customer arrive and/or departs from the system during a given time unit [7,8]. An exact system can only be modeled according to the occurrence of an arriving customer in a given slot at some observing point. Hence, discrete time queueing systems can be modeled in two ways based on the nature of arriving customer whether it come at the start (beginning) of a slot or at the end of a slot, called as an Early Arrival System or Late Arrival System, respectively [9,10].

Phase-type distributions are based on the method of stages technique and it consists of a general mixture of exponentials and is characterized by a finite and absorbing Markov chain [11]. The number η of phases in the PH distribution is equal to the number of transient states in the associated (underlying) Markov chain. A PH distribution represents random variables that are measured by the time X that the underlying Markov chain spends in its transient portion till absorption. Phase type distribution can be used in discrete time queueing system to model the system. In discrete time phase type distribution, stages are inter-related with geometric distribution. We use discrete time hypogeometrical distribution to model the arrival process of a load management discrete time queueing system with dual service rate [12].

In this paper, we develop and demonstrate a flow time process and service process transition diagram to analyze the flow time of customers in a load management late arrival discrete time queueing system with dual service rate.

The organization of this paper is as follows: Discrete time hypogeometric distribution, flow time calculation and system model are briefly discussed in Section 2. In Section 3, load management late arrival discrete time queueing system with dual service rate and its system state transition diagram, Flow process transition diagram and its starting probabilities are discussed. Finally the results and conclusions are presents in Sections 4 and 5 respectively.

2. TERMINOLOGY AND MODELS

In this section we first define the terminology used in this paper. Next, the considered system model is elaborated.

2.1 DISCRETE TIME HYPOGEOMETRIC DISTRIBUTION

Discrete-time hypogeometrical distribution defined as the phases in series with different geometrical rates (α_1, α_2, ...) as shown in Fig. 1. Hypogeometric random variable arises when a data unit passes through several phases/stages with geometrically distributed sojourn times. Let the arrivals be in two independent exponential stages with rates α_1 and α_2. The flow time through such networks has a geometric distribution.

A hypogeometric random variable X with parameter α_1 and α_2 has the probability mass function of:

$$P(X = n) = \sum_{k=1}^{n-1} (1-\alpha_1)^{k-1} \alpha_1 (1-\alpha_2)^{n-k-1} \alpha_2$$

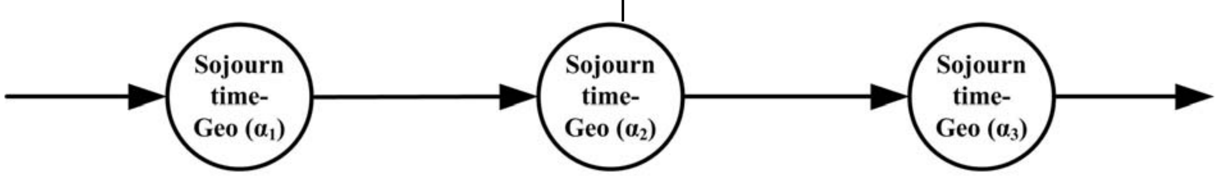

FIG. 1. DISCRETE-TIME HYPOGEOMETRICAL DISTRIBUTION WITH PARAMETERS α_1, α_2 AND α_3

The mean of the distribution is given as:

$$E(X)=\sum_{n=1}^{\infty} nP(X=n)$$

and the variance of the distribution can be determined from the following equation.

$$Var(X)=\sum_{n=1}^{\infty}(n-E(X))^2 P(X=n)$$

3.1 Flow Time Calculation

The time between the entrance and departure of the customer in the system is called flow time. The combination of the system state and service process Markov chains is called the flow process Markov chain which describes the flow of the test customer. The time Markov chain required to flow from the state i to the absorbing state is defined by the complementary cumulative distribution function $\varphi_i(.)$ and defined as:

$$\varphi_1(n)=P\left(X_n \notin \mathrm{H} \middle| X_0=i\right)$$

Through the recursion

$$\varphi_1(n-1)=\sum_k p_{ik}\varphi_k(n)$$

and

$$\varphi_i(0)=0 \qquad i\in \mathrm{H}$$
$$=1 \qquad i\notin \mathrm{H}$$

The probability mass function can be determines as:

$$P(\Theta=n)=P(\Phi\succ n-1)-P(\Phi\succ n)$$

2.3 System Model

The discrete time finite queue with two thresholds is shown in Fig. 2. The discrete time hypogeometric distribution with two phases is used to model the arrival process and service process follows the geometric distribution. The probability of service depends on the two threshold of the queue. If the number of customers in the queue not reaches to the threshold S_{fast}, the probability of the service is β_1. If the queue size reached to the threshold S_{fast}, the probability of service change to the β_2 and then the probability of service back to the β_1, when queue size decreased to the S_{normal}.

3. LOAD MANAGEMENT LATE ARRIVAL DISCRETE TIME QUEUEING SYSTEM WITH DUAL SERVICE RATE

In order to understand the concept of load management late arrival discrete time queueing system with dual service rate we first explore the system state transitions. Using state transition diagram we develop the flow process for load management and develop their starting probabilities. The system transition diagram is used to represent the overall system behavior. Flow process is more efficient to show the flow of customer in the system from its entrance and leaving the system and the starting probabilities shows the all possible states where customer is allow entering the system.

3.1 System State Transition Diagram

The required transition diagram of the queue is constructed as shown in Fig. 3. The system is modeled as a late arrival system and arrival follows a two phases to enter in the system. The system state represented by a $\pi_{i,j}$, where i represent the number of customers in the

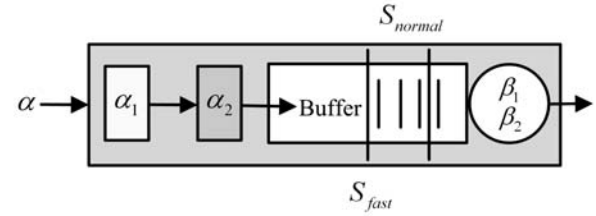

FIG. 2: QUEUEING MODEL

system and j represents the phase of the arrival, and the system states $\pi_{0,1}$ and $\pi_{0,2}$ denotes an idle state of the system. The system state $\pi_{4,1}$ represents a system state when a system reaches a threshold S_{fast}, therefore, in this case the normal probability of service β_1 is changed to the faster probability of service rate, that is, β_2. When the system size reduces and reaches to the system state $\pi_{1,1}$, then the probability of service rate changes to normal service rate, that is, β_1. The system state $\pi_{5,2}$ represents the blocking state of the system, where customers are not allowed to join the queue if no departure event takes place in a given slot but if a customer departs before an arrival event in the same slot only then an arriving customer can join the queue.

3.2 Flow Process

From the concept of calculating the waiting or flow time of a customer in a queueing system in which the offered service rate directly depends on the state of the system at a given time, it is very advantageous to obtain and understand the waiting or flow process. It is defined as the time that is needed to reach a certain state. This technique is also known as life-time process. For instance, to obtain performance measures for the flow time, we consider the doom or outcome of a test customer.

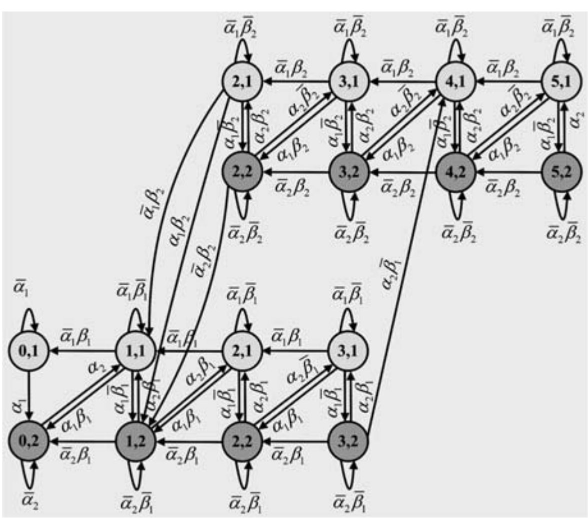

FIG. 3. SYSTEM STATE TRANSITION DIAGRAM

This test customer arrives at an arbitrary time, and if accepted by the system, its flow process will begin. In this way, the outcome of the test customer can be traced, while all possible influences from other customers can be taken into account. The flow process ends with an absorbing state which might be the end of service. Hence, while talking about a system state process, a flow process has a finite duration. The flow process can be considered as a special system state process with absorbing states from which a test customer does not return. It is advantageous to specify the flow process by a flow diagram containing all states a test customer probably visits during its flow process. In addition, if the test customer encounters a full system (including any in service) upon its arrival, it will be rejected, unless there one customer already departed from the system when it arrives. The flow process diagram must be constructed in such a way that it takes into account all customers influencing the outcome of the test customer. The Markov chains of the service process and the flow process are constructed as shown in Fig. 4. In general, it is not possible to determine the time that a customer needs to flow through a queueing system based only on the system state. Instead, the fate of a so called test customer is considered. The test customer enters the queueing system at an arbitrary point of time and begins its flow process. The flow process ends when the test customer has again left the system. The length of the interval between entry and departure of the test customer is the flow time. Only such test customers are considered that are accepted by the queueing system when they arrive. Test customers who are rejected (for example, because the system is full) are ignored. In most cases, the fate of the test customer depends on the system state as well as on its evolution within the service discipline of the system (in the following referred to as service process). Therefore, the Markov chain for the flow process is a combination of the Markov chain for the system state and the Markov chain that describes the service process.

Three kinds of states can be distinguished:

(1) States in which the flow process can start. These states can be taken immediately after the arrival of the test customer.

(20 States in which the flow process cannot start. These states can be reached only indirectly.

(3) States in which the test customer has left the queueing system. These states are absorbing, because when such a state is reached, the flow process has ended.

Depending on in which state the queueing system is when the test customer arrives and when the flow process begins, the flow time will be longer or shorter.

3.3 Starting Probabilities of Flow Process

The starting states probabilities where test customer can start its flow process are:

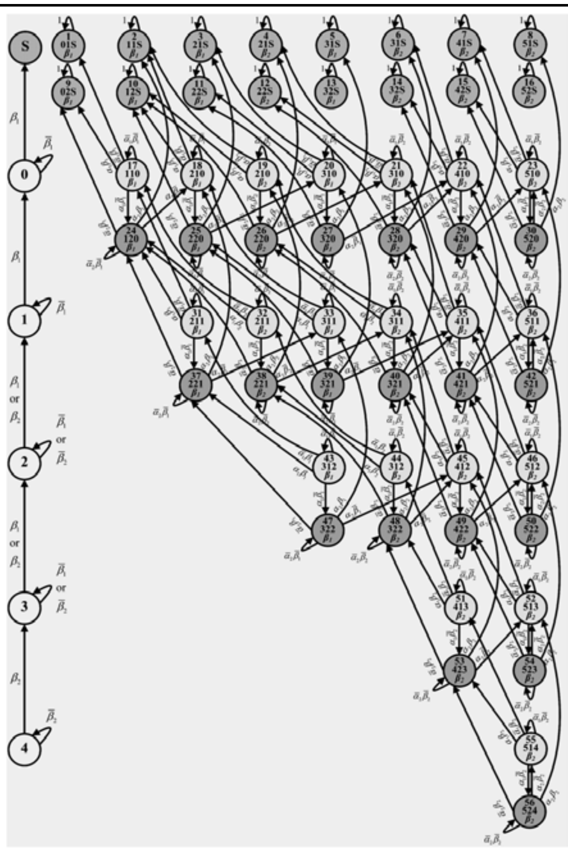

FIG. 4. SERVICE AND FLOW PROCESS TRANSITION DIAGRAM

P(Starting State 1/0)

$$= \sigma^{\phi}_{(110)}$$

$$= \left[\frac{\pi_2}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}} + \frac{\pi_4}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_1\right]$$

$$\times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_1$$

P(Starting State 2/1)

$$= \sigma^{\phi}_{(211\beta_1)}$$

$$= \left[\frac{\pi_4}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_1 + \frac{\pi_6}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_1\right.$$

$$\left. + \frac{\pi_8}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_2\right] \times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

P(Starting State 2/1)

$$= \sigma^{\phi}_{(211\beta_2)}$$

$$= \left[\frac{\pi_8}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_2\right] \times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

P(Starting State 3/2)

$$= \sigma^{\phi}_{(312\beta_2)}$$

$$= \left[\frac{\pi_6}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_1 + \frac{\pi_{10}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_1\right]$$

$$\times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

P(Starting State 3/2)

$$= \sigma^{\phi}_{(3122\beta_2)}$$

$$= \left[\frac{\pi_8}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_2 + \frac{\pi_{12}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_2\right]$$

$$\times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

P(Starting State 4/3)

$$= \sigma^{\phi}_{(413\beta_2)}$$

$$= \left[\frac{\pi_{10}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_1 + \frac{\pi_{12}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_2\right.$$

$$\left. + \frac{\pi_{14}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_2\right] \times \frac{1}{1-\frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

P(Starting State 5/4)

$$= \sigma\left(^{\phi}_{514\beta_2}\right)$$

$$= \left[\frac{\pi_{14}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\bar{\beta}_2 + \frac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}\beta_2\right]$$

$$\times \frac{1}{1-\dfrac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

Where as, from above Equations of starting probabilities following factor occur.

$$\frac{1}{1-\dfrac{\pi_{16}}{\pi_2+\pi_4+\pi_6+\pi_8+\pi_{10}+\pi_{12}+\pi_{14}+\pi_{16}}}\bar{\beta}_2$$

This is due to the fact that a test customer cannot begin its flow process if the queueing system is already full on its arrival.

4. RESULTS

The analytical and simulation programs for calculating the pmf of each starting state and total pmf are developed in visual C++ and Matlab.

The analytical results for pmf of each starting state where a test customer can enter into the system and starts its flow process are shown in Fig. 5(a-b) for various values of α_1, α_2, β_1 and β_2. In these figures various system-states are shown from which a test customer can enter and begin its flow until it reaches to an absorption state (departs from the system). These system-states are marked as pmf 17, 31,32, 43, 44, 51 and 55, which actually refer and represent all those possible system-states from which a test customer can enter as shown in Fig. 4. Also in Fig. 5(a-b), total

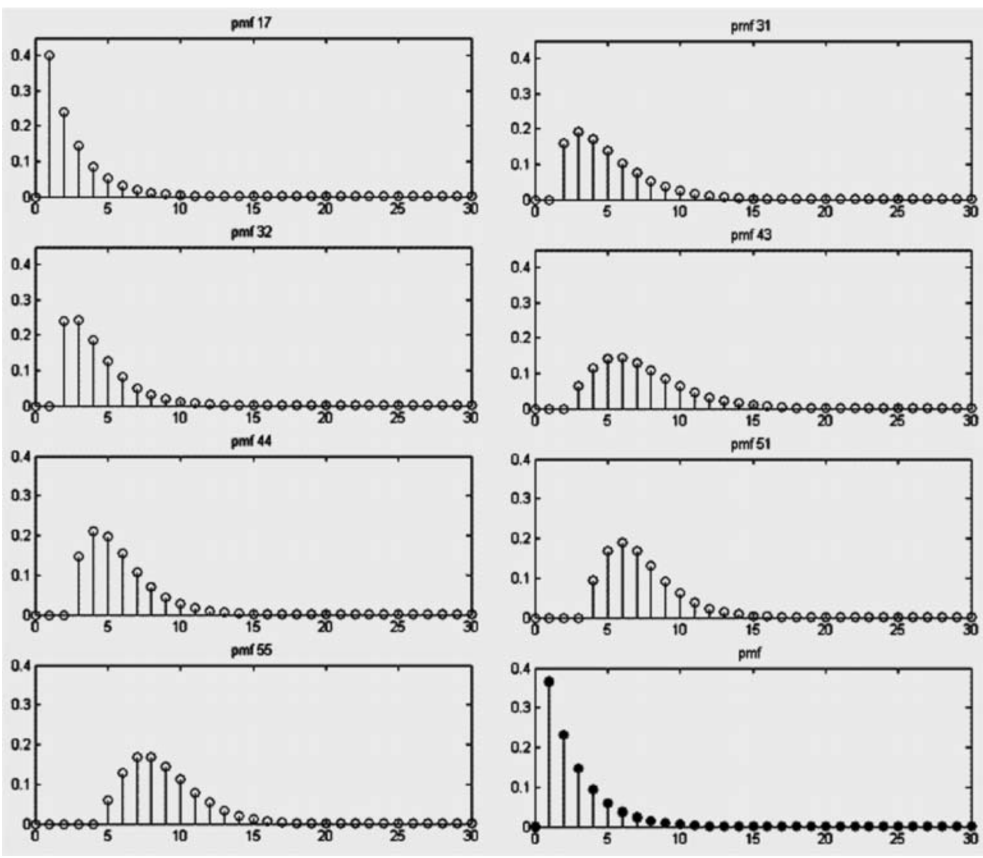

FIG. 5(a). PROBABILITY MASS FUNCTION OF VARIOUS STARTING STATES AND TOTAL PROBABILITY MASS FUNCTION OF LOAD MANAGEMENT DISCRETE TIME QUEUEING SYSTEM WITH DUAL SERVICE RATE, LATE ARRIVAL SYSTEM, WHEN α_1=0.05, α_2=0.2, β_1=0.3 AND β_2=0.5 FILLED STEMS: SIMULATION RESULTS. UNFILLED STEMS: ANALYTICAL RESULTS probability mass functions are shown. The obtained analytical results are validated with the simulation results which show exact matching. In Figs. 4-5(a-b), filled stems show the simulation results whereas unfilled stems show analytical results. Both of them are exactly overlapped for different values of α_1, α_2, β_1 and β_2.

5. CONCLUSIONS

In this paper, we used a discrete time queueing approach to model the load anagement system with dual service rate. The system is solved using late arrival system in which the arrival process is modeled through a discrete time hypogeometric phase type distribution. The transition diagrams of flow process and service process are constructed to trace the flow time of customers in the system. The probability mass function pmf of each starting state and total pmf was calculated and results were also presented. Obtained analytical results are also validated with simulation results for numerous values of α and β which show exact resemblance.

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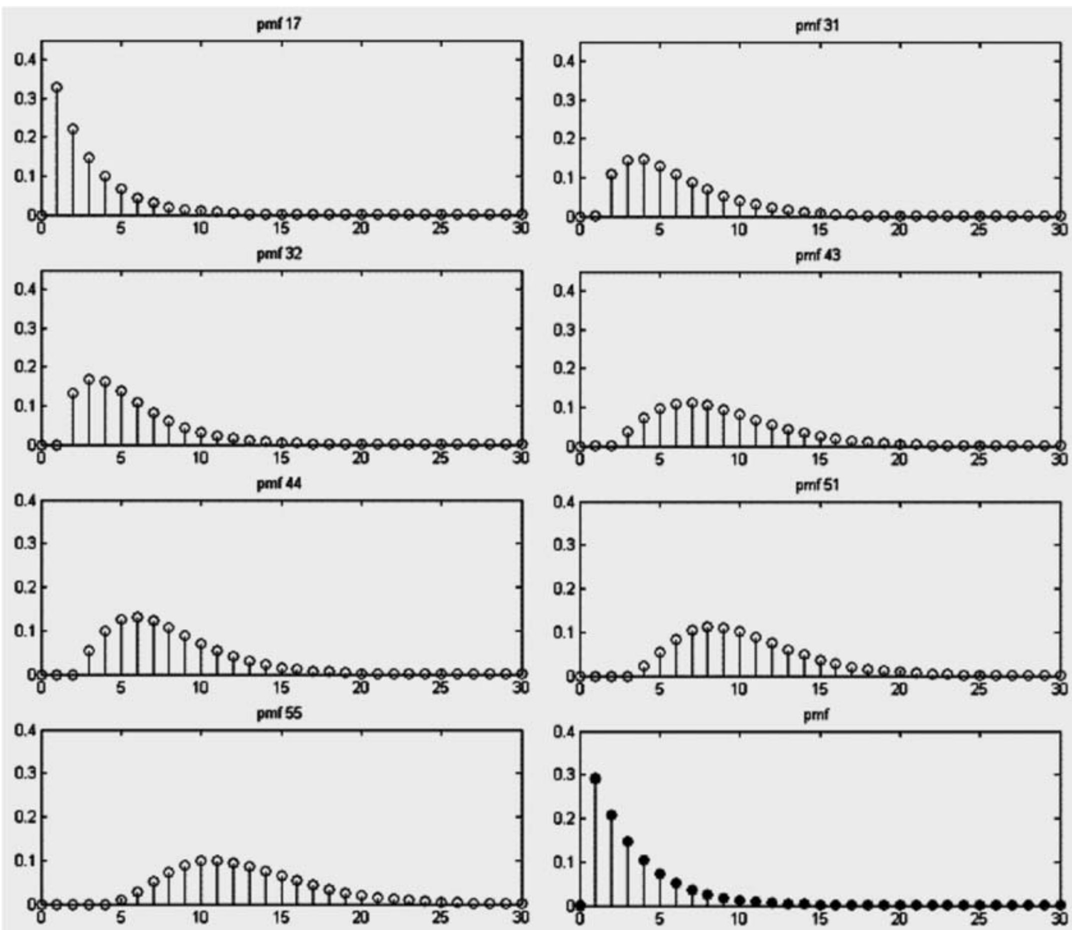

FIG. 5(b). PROBABILITY MASS FUNCTION OF VARIOUS STARTING STATES AND TOTAL PROBABILITY MASS FUNCTION OF LOAD MANAGEMENT DISCRETE TIME QUEUEING SYSTEM WITH DUAL SERVICE RATE, LATE ARRIVAL SYSTEM, WHEN α_1=0.05, α_2=0.2, β_1=0.3 AND β_2=0.5 (b) α_1=0.2, α_2=0.3, β_1=0.4 ANDβ_2=0.6 FILLED STEMS: SIMULATION RESULTS. UNFILLED STEMS: ANALYTICAL RESULTS

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